From Inheritance Relation to Non-Axiomatic Logic

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ABSTRACT

Non-Axiomatic Reasoning System is an adaptive system that works with insufficient knowledge and resources.

At the beginning of the paper, three binary term logics are defined. The first is based only on an inheritance relation. The second and the third suggest a novel way to process extension and intension, and they also have interesting relations with Aristotle's syllogistic logic.

Based on the three simple systems, a Non-Axiomatic Logic is defined. It has a term-oriented language and an experience-grounded semantics. It can uniformly represents and processes randomness, fuzziness, and ignorance. It can also uniformly carries out deduction, abduction, induction, and revision.

KEYWORDS: **Insufficient knowledge and resources, non-axiomatic logic and reasoning system, term logic, experience-grounded semantics, measurements of uncertainty, revision, deduction, abduction, induction.**

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1. INTRODUCTION

Non-Axiomatic Reasoning System (NARS) is proposed as a formal model of intelligent reasoning systems [40].

A reasoning system, in its general form, has the following components [5, 34]:

1. A domain-independent *formal language* by which the system can communicate with its environment, that is, to get knowledge and questions, and to provide answers according to its knowledge;
2. A *semantics* that shows (in principle) how to understand and determine the meaning of the terms and the truth value of the sentences in the language;
3. A set of formal *inference rules* which generate valid (according to the semantics) conclusions from given premises;
4. A *memory*, which serves as a storage of knowledge and questions, and a working place as well;
5. A *control mechanism* which chooses premises and rule(s) in each inference step to answer the questions.

The first three components can be called a "logic", in the broad sense of this term [37].

As an intelligent system, NARS is designed to be an adaptive system under the constraints that its knowledge and resources are usually insufficient to answer the questions proposed by the environment [40, 42]. Concretely, it has the following features:

Finite: The system works with respect to its constant information processing capacity;

Real-time: New knowledge and questions can arrive at any time, and questions have time requirements attached;

Open: No constraint is put on the knowledge and questions that the system can encounter, as long as they are expressible in the formal language;

Adaptive: The system self-improves its behaviors under the assumption that its future experience will be similar to its past experience, where the "experience" of the system is indicated by its history of communication with its environment.

What follows from the above requirements is: the system need to represent and process various types of uncertainty, and to generate plausible answers according to its experience. Therefore, the traditional logic systems, such as first-order predicate logic, cannot be applied. Though there are already many approaches that are designed to deal with uncertain or incomplete knowledge, none of them is completely based on the "insufficiency of knowledge and resources" assumption, defined by the above features.

This paper can be seen as the first step of the NARS project. In the following sections, we start by defining an *inheritance relation*, and then three simple logics based on it. They provide a formal foundation on which the

simplest *Non-Axiomatic Logic*, NAL1, can be defined. NAL1 handles multiple types of uncertainty (such as randomness, fuzziness, ignorance, and so on), as well as multiple types of inferences (such as revision, deduction, abduction, induction, and so on), in a unified manner.

2. THREE SIMPLE SYSTEMS

2.1. Inheritance Logic

The four logics discussed in this paper are all *term logics*, which are different from *predicate logics* by having the following features: [4, 13]

1. Each proposition consists of a *subject term* and a *predicate term*, which are related by a *copula*;
2. The copula is intuitively interpreted as "to be";
3. The basic inference rules take two propositions that share a common term as premises, and get a conclusion from them, in which the other two (unshared) terms are related by a copula.

In the simplest case, there is only one type of copula in the system, and all the terms are "atomic", that is, has no internal structure. In this way, we get an *Inheritance Logic* (IL).

DEFINITION 2.1. *A* term *is a string of letters in an alphabet.*

DEFINITION 2.2. *The* inheritance relation, "$\sqsubset$", *is a reflexive and transitive relation between two terms.*

DEFINITION 2.3. *A* proposition *consists of two terms related by the inheritance relation. In a proposition "$S \sqsubset P$", S is the* subject term, *and P is the* predicate term, *of the proposition.*

The intuitive meaning of the inheritance relation is closely related to many well-known relations, for instance, "ISA" (in semantic network), "belongs to" (in Aristotle's syllogism), "subset" (in set theory), "inheritance assertion" (in inheritance network [35]), as well as many relations studied in psychology and philosophy, such as "type-token", "category-instance", "general-specific", and "superordinate-subordinate" [7]. What make it different from the others are: it is a relation between two *terms*, and the

relation is completely defined by the two properties: *reflexivity* and *transitivity*. [1]

This logic (as well as the following two) can be interpreted in the usual model-theoretic way. Terms have no meaning until a model is set up, where they are mapped into objects in a domain, and the inheritance relation is also mapped into a (reflexive and transitive) relation in the domain. Such a mapping give the terms "meaning". A proposition is "true" if and only if the two corresponding objects really have the relation in the domain.

Under such an interpretation, "valid inference rules" are the rules that produce true conclusions from true premises. Obviously, there are exactly two valid rules, corresponding to reflexivity and transitivity, respectively.

With these two rules, from any non-empty and finite set of propositions K (as premises), the following algorithm can generate the set of all valid conclusions K^*:

1. Let $K^* = K$;
2. For each term T appearing in K^*, put "$T \sqsubset T$" in K (if it is not already there);
3. For each pair of propositions "$S \sqsubset M$" and "$M \sqsubset P$" in K^*, put "$S \sqsubset P$" in K^* (if it is not already there).

When IL is implemented in a reasoning system, two types of questions can be answered according to a given K (as premises):

evaluation: "$S \sqsubset P$", that is, "Is there an inheritance relation from S to P?";

selection: "$? \sqsubset P$" (or "$S \sqsubset ?$"), that is, "Which term has an inheritance relation to P (or from S)?"

To answer the questions, the system can simply generate K^*, then search for a proposition with the form "$S \sqsubset P$".

Obviously, both types of question are decidable. For the evaluation question, the system will answer "Yes" when "$S \sqsubset P$" is in K^*, otherwise, "No" (so the system works under the "closed-world assumption"); for the selection questions, the system will answer "X" when there is a X that has the desired relation with S (or P). X cannot be S (or P) itself — that will be trivial, and if there is more than one answers, any one of them is fine. If there is no such a term, the answer is "No".

The system can also be described in terms of a network. K can be represented by a directed graph, where terms are nodes and inheritance relations are directed links (say, from the subject to the predicate). The

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[1] The membership relation "$\in$" cannot be represented in IL, though it can be introduced in the extensions of IL [40]. Therefore, the subject of a proposition cannot be a "singular term", such as "Tweety" or "Socrates". On the other hand, as in Aristotle's logic, "the same term may be used as a subject and as a predicate without any restriction" [24].

questions are search problems either for the existence of a *path* from a given node to another given node (evaluation) or for a *node* in a path from (or to) a given node (selection).

Up to now, we have got a complete reasoning system. Although IL is quite simple (even trivial) by itself, we will see that this logic provides a solid ground for its successors.

2.2. Extension and Intension

In IL, the *extension* and *intension* of a term is defined, relative to a set of propositions K, as the following:

DEFINITION 2.4. *The* extension *of a term T is a set of terms $E_T = \{x \mid (x \sqsubset T) \in K^*\}$. The* intension *of T is a set of terms $I_T = \{x \mid (T \sqsubset x) \in K^*\}$.* [2]

This definition is of great importance. Traditionally, *extension* and *intension* refer to two aspects in the meaning of a term: its *instances* and its *properties*. Extension is usually defined as an *object*, or a set of objects, which is in a "physical world", and denoted by the term; intension is usually defined as a *concept*, which is in a "Platonic world", and denoting the term [4, 19]. In spite of the differences among the exact ways the two words are used by different authors, they indicate relations between a term in a language and something *outside* the language. However, in the current theory, they are defined as (the two sides of) a relation between two terms, which is *within* the language, and the definition still keeps the intuitive senses that extension refers to "instances", and intension refers to "properties". We can see why such a definition is appropriate when the major result of the paper, NAL1, is discussed.

Now let us simply accept the definition and its implications, which are:

1. Extension and intension are defined in such a symmetric way that for any result about one of them, there is a dual result about the other.
2. Each proposition reveals part of the intension for the subject term and part of the extension for the predicate term.
3. Since the inheritance relation is reflexive, all terms have a non-empty extension and a non-empty intension — at least they have the term itself in it.

From the definition, it is not difficult to get the following results (where "$\Longleftrightarrow$" means "if and only if"):

$$(S \sqsubset P) \Longleftrightarrow (E_S \subseteq E_P) \Longleftrightarrow (I_P \subseteq I_S),$$

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[2] The extension and intension of a term are ordinary sets, but a term itself is not a set.

$$(E_S = E_P) \Longleftrightarrow (I_S = I_P).$$

The first one means "There is an inheritance relation from S to P" is identical with "S inherits P's intension" and "P inherits S's extension". This is the reason that "$\sqsubset$" is called "inheritance relation". [3] The second one shows that the extension and intension of a term are mutually determined. Therefore, given one of them, the other can be uniquely obtained.

2.3.   Extensional Term Logic

In IL, we only distinguish two types of relation between the extensions of two terms: whether one is completely included in the other. Now let's consider other possible binary relations between the extensions of two terms S and P.

As sets, if $E_S \subseteq E_P$ (or $E_P \subseteq E_S$), then such a *complete* and *affirmative* inheritance of extension can be represented and processed by IL, defined as above. How about *partial* or *negative* inheritance relations between the extensions of two terms? A successor of IL, *Extensional Term Logic* (ETL), is defined to capture these relations.

By introducing four types of copulas, the following definition naturally extends the "$\sqsubset$" relation defined in IL:

DEFINITION 2.5.

$$S \sqsubset_a P \quad \textit{if and only if} \quad E_S - E_P = \emptyset$$

$$S \sqsubset_e P \quad \textit{if and only if} \quad E_S \wedge E_P = \emptyset$$

$$S \sqsubset_i P \quad \textit{if and only if} \quad E_S \wedge E_P \neq \emptyset$$

$$S \sqsubset_o P \quad \textit{if and only if} \quad E_S - E_P \neq \emptyset$$

They can be understood as "All S are P", "No S is P", "Some S are P", and "Some S are not P", respectively.

Since $(E_S - E_P = \emptyset) \Longleftrightarrow (E_S \subseteq E_P)$, we have $(S \sqsubset_a P) \Longleftrightarrow (S \sqsubset P)$.

Though described differently, ETL turns out to be isomorphic with Aristotle's syllogistic logic. For each property of Aristotle's logic [1, 24, 28], there is a corresponding one in ETL, and *vice versa*.

The square of opposition: The relations among the four types of extensional inheritance can be represented in Figure 1 [4], where there are four types of relations:

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[3] Here "inheritance" is used for a *logical* relation between two terms, rather than an idea about the *implementation* of a knowledge base, by which storage space can be saved [7, 35].

1. If "$S \sqsubset_a P$" is true, then "$S \sqsubset_i P$" is true; if "$S \sqsubset_e P$" is true, then "$S \sqsubset_o P$" is true.
2. "$S \sqsubset_a P$" and "$S \sqsubset_o P$" must be one true and one false; "$S \sqsubset_e P$" and "$S \sqsubset_i P$" must be one true and one false.
3. "$S \sqsubset_a P$" and "$S \sqsubset_e P$" cannot both be true.
4. "$S \sqsubset_i P$" and "$S \sqsubset_o P$" cannot both be false.

Conversion: If "$S \sqsubset_e P$" is true, so is "$P \sqsubset_e S$"; if "$S \sqsubset_i P$" is true, so is "$P \sqsubset_i S$".

Syllogisms: For each valid syllogism in Aristotle's logic, there is a corresponding (valid) inference rule in ETL. All of the rules are listed in Table 1.

All of these properties can be proven from the definitions by reasoning according to set theory.

Although the four types of proposition are defined in terms of extensions, which is defined by the inheritance relation "$\sqsubset$", the above rules make ETL directly applicable to a set of "extensional propositions" (of the four types), and use "$\sqsubset$" only for the purpose of interpretation. In this way, IL is both a meta-system of ETL (because the former is used to define the basic components of the latter) and a sub-system of ETL (because the "$\sqsubset$" relation is identical to the "$\sqsubset_a$" relation).

When implemented in a reasoning system to solve domain problems, the premises must be consistent. It is possible for the system itself to determine whether a given finite set of propositions is consistent by exhausting its implications, then checking for the second and third type of relations in the Square of Opposition.

Compared with IL, the questions that ETL can answer are more complicated:

evaluation: "$S \;?\; P$" now have five possible answers, corresponding to the four relations and "I don't know" (undetermined), respectively. If both "$S \sqsubset_a P$" and "$S \sqsubset_i P$" (or "$S \sqsubset_e P$" and "$S \sqsubset_o P$") are got, the former is reported as the result; If both "$S \sqsubset_i P$" and "$S \sqsubset_o P$" are got, they are reported together.

selection: The four different relations correspond to four kinds of questions. When there are multiple answers, any one is equally good as the result. When no such a term can be found, "No" is the reply.

The questions are still decidable, because an answer will be provided in finite time for every question.

The system can be described in terms of network, too. Now there are four different types of links in the network, corresponding to the four kinds of inheritance relations.

If only the relations "$S \sqsubset_a P$" and "$S \sqsubset_e P$" are represented and processed, ETL will degenerate into a special case, which is identical with the

"Monotonic Inheritance Network" defined in [10].

2.4. Intensional Term Logic

Since extension and intension are defined as a "dual" in IL, we get an Intensional Term Logic (ITL) "for free", which is isomorphic with ETL.

DEFINITION 2.6.

$$S \sqsubset^a P \quad \text{if and only if } I_P - I_S = \emptyset$$

$$S \sqsubset^e P \quad \text{if and only if } I_P \wedge I_S = \emptyset$$

$$S \sqsubset^i P \quad \text{if and only if } I_P \wedge I_S \neq \emptyset$$

$$S \sqsubset^o P \quad \text{if and only if } I_P - I_S \neq \emptyset$$

These propositions can be interpreted intuitively as "S has all of the properties of P", "S has none of the properties of P", "S has some of the properties of P", and "S lacks some of the properties of P", respectively. Here, the quantifiers are applied to the properties (intension) of the predicate, rather than to the instances (extension) of the subject (as Aristotle did) or predicate (as Bentham and Hamilton did, see [4]).

The propositions represented in this way are closely related to "typicalness" [31], "representativeness" [36], "normality" [20], and "fuzziness" [45]. All these concepts are proposed, from different standing points, to capture the phenomenon that an instance does (or doesn't) possess all (or some) properties of a category. The related problems cannot be properly represented and processed by any extensional logic.

ITL has isomorphic properties with ETL, such as $(S \sqsubset^a P) \Longleftrightarrow (S \sqsubset P)$, the same Square of Oppositions, and identical conversion rules for the "$\sqsubset^e$" and "$\sqsubset^i$" relations. As a result, we know that $(S \sqsubset^a P) \Longleftrightarrow (S \sqsubset_a P)$, as well as $(S \sqsubset^o P) \Longleftrightarrow (S \sqsubset_o P)$. However, "$S \sqsubset^e P$" and "$S \sqsubset_e P$" are different, so are "$S \sqsubset^i P$" and "$S \sqsubset_i P$".

The inference rule table of ITL, Table 2, is isomorphic to that of ETL (they are symmetric to the main diagonal).

Though nothing new technically, ITL suggests a simple, and psychologically plausible way to process intensions. Also in this way, extension and intension are naturally related.
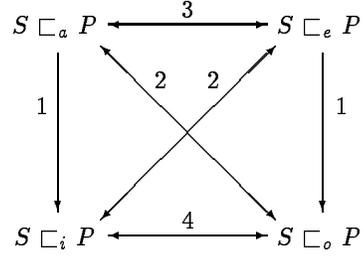
$$S \sqsubset_a P \quad \xleftarrow{\ 3\ } \quad S \sqsubset_e P$$

(vertical arrow labeled 1 on left; vertical arrow labeled 1 on right; diagonal arrows labeled 2 and 2)

$$S \sqsubset_i P \quad \xleftarrow{\ 4\ } \quad S \sqsubset_o P$$

Figure 1. The square of opposition

$J_2 \setminus J_1$	$M \sqsubset_a P$	$M \sqsubset_e P$	$M \sqsubset_i P$	$M \sqsubset_o P$	$P \sqsubset_a M$	$P \sqsubset_e M$	$P \sqsubset_i M$	$P \sqsubset_o M$
$S \sqsubset_a M$	$S \sqsubset_a P$	$S \sqsubset_e P$				$S \sqsubset_e P$		
$S \sqsubset_e M$					$S \sqsubset_e P$			
$S \sqsubset_i M$	$S \sqsubset_i P$	$S \sqsubset_o P$				$S \sqsubset_o P$		
$S \sqsubset_o M$					$S \sqsubset_o P$			
$M \sqsubset_a S$	$S \sqsubset_i P$	$S \sqsubset_o P$	$S \sqsubset_i P$	$S \sqsubset_o P$	$S \sqsubset_i P$	$S \sqsubset_o P$	$S \sqsubset_i P$	
$M \sqsubset_e S$					$S \sqsubset_e P$			
$M \sqsubset_i S$	$S \sqsubset_i P$	$S \sqsubset_o P$				$S \sqsubset_o P$		
$M \sqsubset_o S$								

Table 1. Inference rules of ETL

$J_2 \setminus J_1$	$M \sqsubset^a P$	$M \sqsubset^e P$	$M \sqsubset^i P$	$M \sqsubset^o P$	$P \sqsubset^a M$	$P \sqsubset^e M$	$P \sqsubset^i M$	$P \sqsubset^o M$
$S \sqsubset^a M$	$S \sqsubset^a P$		$S \sqsubset^i P$		$S \sqsubset^i P$		$S \sqsubset^i P$	
$S \sqsubset^e M$	$S \sqsubset^e P$		$S \sqsubset^o P$		$S \sqsubset^o P$		$S \sqsubset^o P$	
$S \sqsubset^i M$					$S \sqsubset^i P$			
$S \sqsubset^o M$					$S \sqsubset^o P$			
$M \sqsubset^a S$		$S \sqsubset^e P$		$S \sqsubset^o P$	$S \sqsubset^i P$	$S \sqsubset^e P$		
$M \sqsubset^e S$	$S \sqsubset^e P$		$S \sqsubset^o P$		$S \sqsubset^o P$		$S \sqsubset^o P$	
$M \sqsubset^i S$					$S \sqsubset^i P$			
$M \sqsubset^o S$								

Table 2. Inference rules of ITL

3.   NON-AXIOMATIC LOGIC 1: SEMANTICS AND SYNTAX

3.1.   Experience-grounded Semantics

Though IL, ETL, and ITL have interesting properties, they are still "axiomatic logics" in the following senses:

1. They use a model-theoretic semantics, by which "true" is defined as "isomorphic with the state of affairs (in a model)";
2. A finite set of premises, whose truthfulness is presupposed by the system, is used as the start point of all inferences;
3. The inference rules are valid in the sense that they only derive true conclusions from true premises;
4. The system answer questions by constructing a proof from the premises to the desired proposition, and the choice of premises and rules in each step is determined by an algorithm;
5. All the resources requirements of the algorithm can be satisfied, and the answer of a question is defined independent to its time-space resources expense.

Axiomatic logics work well in many domains, especially in mathematics, but they require sufficient knowledge and resources with respect to the problems to be solved. Specially, the premises must imply the desired answers, and the system must have enough time-space resources to actually derive out those answers.

What should a system do if the above requirements cannot be satisfied? In the following, a "non-axiomatic logic", NAL1, is defined, which works with insufficient knowledge and resources. Of course, in such a situation no correct answer can be guaranteed. However, if a system have to work in such a situation, then some answers are still better than arbitrary guesses or always saying "I don't know".

Since NARS is designed to be adaptive, it makes judgments based on its experience, although such judgments may conflict with the system's future experience. [4]

In the working environment of NARS, as defined in the introduction of the paper, model-theoretic semantics cannot be applied. If "truth value" is still used to indicate whether (or to what degree) a judgment is mapped exactly into an "objective state of affair", then no judgment can get such a truth value. However, we have reason to argue (though I prefer not to discuss the philosophical issues here) that in everyday life and empirical

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[4]In the first three systems, the name "proposition" is used for a binary assertions. In the last system, the name "judgment" is used for a multi-valued assertions. The former can be seen as a special case of the later.

science, where we do suffer from the insufficiency of knowledge and re-sources, whether (or to what degree) a judgment is "true" is determined by comparing it with the experience of the system (a human being or a scientists community) to see whether (or to what degree) the judgment is *supported* by the experience.

If this is the case, can we simply give up the idea of truth value, and label each judgments concretely with its positive (confirmative) and negative (refutive) evidence in the system's experience? This is impossible because the system may not have the resources to do it, and more importantly, the evidence need to be summarized for various operations. To summarize information about evidence into truth values causes information loss, but it is absolutely necessary for the system, because qualitatively different evidence need to be treated in a unified manner [26].

This will lead to what I call "experience-grounded semantics", where *the truth value of a judgment indicates the degree to which the judgment is supported by the system's experience*. Defined in this way, truth value is system-dependent and time-dependent. Different systems may have con-flicting opinions, due to their different experiences. Even for the same system, truth values of judgments are constantly re-evaluated according to new experience.

Concretely, let us assume the experience of the system is represented by a sequence of input judgments $E = < J_1, J_2, \ldots, J_n >$, where all the judgments are sentences of the communication language $\mathbf{L}$, and their truth values are determined in the interaction of the system and the environment. They are used as primary premises by the system. [5]

Comparing E with the premises set K defined in the previous axiomatic logics, we can see two fundamental differences:

1. K is constant, but E changes from time to time;
2. K must be consistent, but E is not necessarily, that is, there may be a pair of judgments J_i and J_j, which assign different truth values to the same relation.

Someone may suggest to treat E as an ordinary "knowledge base". When each piece of new knowledge arrives, the previous judgments that conflict with it are removed. Such a method assumes the system always have suf-ficient resources to find all conflicts among judgments, and new knowledge is always superior to old knowledge. However, the first assumption can-not be accepted by NARS, and so do the second one — "old" and "new" knowledge may come from distinct sources, therefore we cannot treat all conflicts as "updating".

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[5] Accurately speaking, the experience of a system also includes questions asked by the environment. Though the questions influence the resource distribution of the system, they do not effect the evaluation of a truth value, therefore they are ignored here.

What makes the things more complicated is the fact that, also because of the insufficiency of resources, it is impossible for the system to take the complete E into account when evaluating the truth value of a judgment. As a result, truth values are usually determined according to a *section* of E. It is another source of conflicts among the judgments. Because each judgment is only supported by partial experience of the system, judgments summarized from different sections of the experience may conflict with each other.

It is possible, in theory, to use binary logic in this situation. For example, we can distinguish four types of judgments, based on the given experience E, by whether it has positive and negative evidence (as in ETL and ITL). However, such a logic is too weak. For the system to be adaptive, the *weight* (or amount) of evidence is important. A judgment that has be confirmed one thousand times should definitely win a competition with a judgment that has been confirmed only once. When a judgment has both positive and negative evidence, a quantitative comparison is necessary. Otherwise, the system will be unable to make a choice among competing answers for a question — they may be all marked as "possible".

Now, since the truth value of a judgment is actually a measurement of the positive and negative evidence, then what we need to do is:

1. to concretely define the positive and negative evidence for a judgment, and
2. to define a measurement unit.

These problems are hard for predicate logics (as revealed by the famous "Raven Paradox" of Hempel [8]), but in a term logic like NAL1, we can find a natural solution of them.

In summary, the truth value of a judgment in NAL1 is a numerical representation indicating the weights of positive and negative evidence, according (part of) the experience of the system. However, experience is nothing else but a string of judgments, with their truth values, too. So, we seems define a truth value circularly by other truth values.

This is why we first introduced a simple system IL in the previous section. We will build NAL1 as an extension of IL, and use propositions in IL to construct an "ideal experience" for each judgment in NAL1, so to break the definition circle.

Alike in IL, ETL, and ITL, in NAL1 what is represented in each judgment is still "to what extent one term can be used as another", or the inheritance relation between the extensions and intensions of two terms. However, as discussed before, the a, e, i, o relations in ETL and ITL need to be extended to a numerical measurement. For this purpose, a new inheritance relation "$\subset$" is introduced as a refined version of "$\sqsubset$", and a judgment "$S \subset P$" still indicates that "S inherits P's intension, and P inherits S's extension".

For such a judgment, what is its positive or negative evidence? Obviously, if (according to the experience of the system) there is a term T in the extensions (or intensions) of both S and P, then it provides positive evidence for the judgment; on the other hand, if there is a term T in the extension of S, but not in the extension of P, or it is in the intension of P, but not in the intension of S, then it provides negative evidence for the judgment. According to the practice of statistics, the most natural way to calculate the weight of evidence is to simply *count* such terms. In this way, *weight of evidence* take its values in $[0, \infty]$, and is additive when combining two pieces of evidence from distinct sources [32, 41].

If the experience of the system is represented by a string of propositions in IL, then, by using the string as premises set K, we can determine extension and intension for each term. For each judgment "$S \subset P$" in NAL, its truth value can be represented by two of the three weights of evidence: w^+ (positive evidence), w^- (negative evidence), and w (total evidence).

DEFINITION 3.7.

$$w^+ = |E_S \cap E_P| + |I_P \cap I_S|,$$

$$w^- = |E_S - E_P| + |I_P - I_S|,$$

$$w = w^+ + w^- = |E_S| + |I_P|.$$

Though any two of the three can do the job, we will use the $\{w^+, w\}$ pair in the following as the truth value of a judgment.

It must be emphasized that what we provide here is not a method for practically calculate the truth value for every judgment, but a way to *explain* that a truth value *means*. For a reasoning system, the truth values of input judgments are usually determined by the environment (which may be a human user, another computer system, or a sensory device), and the truth values of derived judgments are recursively determined by inference rules from the judgments used as premises. No matter how a truth value is practically generated, we need a unified *interpretation* for it, so to make the numbers understandable, and the inference rules justifiable.

In NAL1, a judgment "$S \subset P\ \{w^+, w\}$" can be generated by many different experience sections, but it is always explained as "The judgment is *as true as* it has been checked w times (by testing terms in S's extension or in P's intension), and the inheritance relation holds in w^+ of them." In this way, we get a measurement that can be uniformly applied to extentional evidence and intensional evidence. To consider more general cases, w and w^+ are not necessarily integers, but they must satisfy $w \geq w^+ \geq 0$.

14

Because the system is always open to new knowledge, w and w^+ have no upper bounds. When $w^+ = w$ and $w \to \infty$, "$S \subset P$ $\{w^+, w\}$" will infinitely approach "$S \sqsubset P$" — that is, there is no, and will no, negative evidence for the inheritance relation. Therefore, "$\subset$" is a "weak version" of "$\sqsubset$", where the latter representing a highly idealized inheritance relation between two terms in a closed world, while the former is based on limited subjective experience of an open system.

Now it is the time to review what we have done: the simple system IL is introduced to make NAL1 easily understandable. Propositions in IL are so simple that they are "self-evident", then, a judgment in NAL1 is explained as a *summary* of a section of "ideal experience", represented by a string of IL propositions, though practically it is not generated in that way.

3.2. Measurements of Uncertainty

As truth value, weight of evidence is also a measurement of uncertainty. Though in principle all the information that we want to put into a truth value is representable in the $\{w^+, w\}$ pair, it is not always natural or convenient for the purpose of NARS. Instead of using "absolute measurements", we often prefer "relative measurements" of uncertainty, such as real numbers in $[0, 1]$. Fortunately, it is easy to define them as functions of weight of evidence.

DEFINITION 3.8. *The* frequency *of a judgment, f, is defined as* $\frac{w^+}{w}$.

f indicates the "success frequency" of the inheritances (of extension and intension) between the two terms, according to the experience of the system. Obviously, this measurement is closely related to probability and statistics, and often appears is our everyday life. However, it is still different from probability under the traditional interpretations (logical, frequentist, and subjective, see [21]) because it is determined by finite empirical evidence.

Another basic difference between probability and frequency is: probability is traditionally interpreted as about extensions of sets. For example, if we say "the probability of '$S \subset P$' is p", it is usually understood as $\frac{|S \cap P|}{|S|} = p$. However, as described earlier, frequency (in NAL1) is about both extensional and intensional relations of the two terms. Therefore, it can be used to process the phenomena like fuzziness, typicalness, and so on. [39] is a detailed description about how to interpret fuzziness and represent it in NARS.

To represent a truth value by a frequency value is not enough for NARS: we still need the information about the absolute value of w to manage the revision of the frequency [38].

Can we find a natural way to represent this information in the form of a

"relative measurements", or specially, as a *ratio*?

An attractive idea is to define it as the "second-order probability". The frequency defined above can be referred to as an estimation of "the first-order probability", and the second-order probability is used to represent how good the estimation is. Actually, several approaches are working along this line [14, 15, 27]. However, there are problems in how to interpret the second value, and how it helps in the related operations [23, 29]. At least under the assumption of insufficient knowledge, it make little sense to talk about the "probability" that "the frequency is an accurate estimation of an 'objective first-order probability' of the inheritance relation". With insufficient knowledge, we not only cannot know whether an inheritance relation will always be kept in the future, but also cannot know its "probability of success", in the sense of "the limit of the experienced frequency". For the same reason, we cannot know how close a given estimation is to the "objective first-order probability", or even whether there is such an "objective probability" as the limit of the frequency. Generally speaking, since the system is open, it is useless to compare the amount of relevant past experience, measured by w, to the future experience, which is (potentially) infinite: the ratio is always 0.

However, it make perfect sense to talk about the "near future". What the system need to know, by keeping the information about w, is how *sensitive* a frequency will be to new evidence, then use the information to make choice among competing judgments. If we limit our attention to a "constant future", we can keep such information in a *ratio* form.

Let us introduce a positive constant k, and say it indicates that by "near future", we mean "to test the inheritance relation for k more times", or identically, "until the weight of the new evidence reaches k". Then we can define a new measurement — confidence.

DEFINITION 3.9. *The* confidence *of a judgment, c, is defined as* $\frac{w}{w+k}$.

Intuitively, confidence is the ratio that the weight of "current relevant evidence" to the weight of "relevant evidence in the near future". It indicates how much the system knows about the inheritance relation, so is similar to Shafer's "reliability" [33] or Yager's "credibility" [44]. Since k is a constant, the more the system knows about the inheritance relation (represented by a bigger w), the more confident the system is about the frequency, since the effect of evidence that comes in the near future will be relatively smaller (we'll see how c actually works in the revision operation in the next section). For our current purpose, k can be any positive number.

Though c is in $[0, 1]$, can be explained as a ratio, and *is* at a higher level than f in the sense that it indicates the stability of f, it cannot

be interpreted as a second-order probability in the sense that it is the probability of the judgment "the (real, or objective) probability of the inheritance relation is f", and cannot be processed in that way according to probability theory. The higher the confidence, the harder the frequency can be changed by new evidence, but this does not mean that the judgment is "truer", or the more "accurate", as some psychologists means by the concept "confidence" [12].

It is easy to calculate w and w^+ from f and c, therefore the truth value of a judgment can also be represented as a pair of ratio $<f, c>$ [38].

Amazingly, there is a third way to represent a truth value in NAL1: as an interval [41]. Let us first define two measurements.

DEFINITION 3.10. *The* lower frequency *of a judgment, l, is defined as* $\frac{w^+}{w+k}$; *the* upper frequency *of a judgment, u, is defined as* $\frac{w^++k}{w+k}$.

Here k is the same constant introduced above. Obviously, no matter what will happen in the *near future*, the "success frequency" will be in the interval $[l, u]$ after the constant period. This is because the current frequency is $\frac{w^+}{w}$, so in the "best" case, when all evidence in the near future is positive, the new frequency will be $\frac{w^++k}{w+k}$; in the "worst" case, when all evidence in the near future is negative, the new frequency will be $\frac{w^+}{w+k}$.

This measurement shares similar intuition with other interval approaches [5, 22, 43]. For example, "ignorance", i, can be represented by the *width* of the interval (here it happens to be $1 - c$, so *ignorance* and *confidence* are complement to each other). However, in NAL1 the interval is defined as the range the frequency will be in the *near* future, rather than in the *infinite* future. In this way, some theoretical problem can be avoided. For example, as discussed about the "second-order probability", it is also impossible for an open system to determine such an interval for the infinite future. So, the interval will be processed differently from those interpreted as "lower/upper bound of (objective) probability". For example, during revision, two intervals that have no common sub-interval still can be combined.

Now we have three functionally identical ways to represent a truth value:

1. as a pair of *weights* $\{w^+, w\}$, where $w \geq w^+ \geq 0$;
2. as a pair of *ratios* $<f, c>$, where $f \in [0, 1]$, and $c \in [0, 1]$; or
3. as an *interval* $[l, u]$, where $0 \leq l \leq u \leq 1$.

Because NARS is designed under the assumption of insufficient knowledge and resources, all the judgments within the system are supported by finite evidence, that is, w is positive and finite. When truth values are represented by the other two forms, this requirement becomes $l < u$, $u - l < 1$, and $0 < c < 1$.

Beyond the normal truth values, there are two limit points useful for the interpretation of truth values and the definition of inference rules:

Null evidence: This is represented by $w = 0$, or $c = 0$, or $u - l = 1$. It means that the system actually know nothing at all about the inheritance relation.

Total evidence: This is represented by $w \to \infty$, or $c = 1$, or $l = u$. It means that the system already know everything about the statement — no future modification of the truth value is possible. Especially, $(S \subset P <1, 1>) \Longleftrightarrow (S \sqsubset P)$.

The one-to-one mappings among the three truth value forms are listed in Table 3.

This table can be easily extended to include w^- (the weight of negative evidence), i ($1 - c$, degree of ignorance), and e (expectation, to be defined in the next section). Actually, any valid (not inconsistent or redundant) assignments to any two of the nine measurements (for examples, $w^+ = 3.5$ and $i = 0.1$, or $f = 0.4$ and $l = 0.3$) will uniquely determine the values of the others. Therefore, the three forms of truth value can even be used in mixture.

To have different, but closely related forms and interpretations for truth value (or uncertainty) has many advantages:

- It give us a better understanding about what a truth value really means in NAL1, since we can explain it in different ways. The mappings also tell us the interesting relations among the various uncertainty measurements.

- It provides a user-friendly interface. if the environment of the system is human users, the uncertainty of a statement can be represented in different forms, such as "I've tested it w times, and in w^+ of them it was true", "Its past success frequency is f, and the confidence is c", or "I'm sure that its success frequency with remain in the interval $[l, u]$ in the near future". Using the mappings in the above table, we can maintain an unique form as internal representation, and translate the other two into it in the interface.

- It make the designing of inference rules easier. For each rule, there should be a function calculating the truth value of the conclusion from the truth values of the premises, and different functions correspond to different rules. As we will see in the next section, for some rule, it is easier to choose a function if we treat the truth values as *weights*, while for another rule, we may prefer to treat them as *ratios* or *intervals*. No matter which form and interpretation is used, the *information* carried is actually the same.

- It is easier to compare the measurements in NAL1 to various other approaches of uncertain representations, because different forms capture different intuitions about uncertainty. See [38, 39, 41] for examples.

3.3. Grammar

As an extension of IL, ETL, and ITL. NAL1 can also answer the two types of questions:

evaluation: To check a given inheritance relation "$S \not\subset P$", it requires a numerical answer, that is, the system evaluates the truth value of the inheritance relation, according to available evidence. Now, "I don't know" corresponds to the special truth value for "Null evidence", as defined before.

selection: To look for a term with an inheritance relation to a given term, "$? \subset P$" (or "$S \subset ?$") need both a term and an evaluation of the truth value of the answer. The system will look for a term T, such that "$T \subset P$" (or "$S \subset T$") has a truth value that make the relation "close" to "$T \sqsubset P$" (or "$S \sqsubset T$"). In the next section, we will define the "closeness" accurately.

In summary, the formal language used by NAL1 is defined by the following grammar:

$$
\begin{aligned}
<judgment> &::= <term>\subset<term><truth\text{-}value> \\
<question> &::= <term>\not\subset <term> \mid ?\subset<term> \mid <term>\subset? \\
<term> &::= <letter> \mid <letter><term> \mid <term>\text{-}<term> \\
<letter> &::= a|b|\cdots|y|z
\end{aligned}
$$

As previously discussed, there are different ways to represent the truth value of a judgment.

For a reasoning system in which NAL1 is implemented, the language can be used both as the "external communication language" (by which the system exchanges information with its environment) and as the "internal representation language" (by which the memory of the system is described).

4. NON-AXIOMATIC LOGIC 1: INFERENCE RULES

4.1. Validity of the Rules

Like other reasoning systems, NARS has a set of inference rules that derive conclusions recursively from input knowledge. However, the traditional definition of validity of inference rules, that is, "to get true conclusions from true premises", no longer makes sense in NARS. With insufficient knowledge and resources, even if the premises are true with respect to the past experience of the system, there is no way to get infallible predictions about the future experience of the system — even the the premises themselves may be challenged by new evidence.

This does not mean that all answers are equally good for a question. As an adaptive system, NARS should answer current questions according to past experience. So, in this situation, an inference rule is *valid* not because the conclusions will not be challenged in the future, but the conclusions are summaries of (and only of) the information carried in the premises, according to the semantics of the system.

A direct implication of the above consequence is that all the inference rules are "local rules", in the sense that each rule only takes a constant number of premises to get conclusions. In NAL1, all rules take one or two judgment(s) as premises. Several authors, for instance Pearl [29], have correctly pointed out that such local rules can cause problems, for examples ignoring related information, repeated using of correlated evidence, and so on. With insufficient resources, ignoring related information is inevitable in each inference step. As long as the system can revise the conclusion when the related information is take into account in a future inference step, this is not a reason to reject local rules. The fact that the system's resources are insufficient itself implies the possibility of ignoring relevant evidence. What we need to do, when designing the system, is not to make sure that all relevant evidence will be considered when a question is answered, but to let the system use relevant evidence as much as possible, under the constraints of affordable resources. The problem caused by correlated evidence is discussed in the next subsection.

4.2. Revision

As discussed above, it is possible (actually it is usually the case) for the judgments in the memory of NARS to conflict with each other, in the sense that at a certain time, there are two co-existent judgments attaching different truth values to the same inheritance relation, as the following:

$$S \subset P \{w_1^+, \, w_1\}$$

$$S \subset P \{w_2^+, \, w_2\}$$

where the truth values are represented as weights of evidence. [6]

Such conflicts are caused by the fact that judgments are based on different sections of the experience of the system, say E_1 and E_2. In principle, we can accurately define the section of experience that a judgment is based on: each input judgment is an atomic section of experience, and a derived judgment is based on the *union* of the experience sections upon which the premises are based.

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[6] Unlike in first order predicate logic, where any conclusion can be derived from a pair of propositions that only differ in there truth values, in NAL a conflict is a local problem that not all results are affected. See the subsection on syllogisms.

According to the semantics of NAL1, as long as E_1 and E_2 has no common elements, the two bodies of evidence supporting the two judgments are not *correlated* to each other (that is, no evidence is repeatedly counted in the two premises). Therefore, the conclusion derived from the two should be

$$S \subset P \{w_1^+ + w_2^+, \ w_1 + w_2\},$$

where the evidence from different sections of experience is summarized, or pooled [41].

This sounds easy, but with insufficient resources, NARS cannot maintain a complete record of related experience for each judgment, because to do this, we have to assume that, no matter how much space is required for the record (the length of an experience section has no upper bound) and how much time is required for the processing, it can always be satisfied.

Therefore the "correlated-evidence recognition problem" cannot be completely solved with insufficient resource. Actually, this is also true for human beings: we simply cannot exactly remember all evidence that supports each judgment we made. On the other hand, the problem must be handled somehow, otherwise, as Pearl said in [29]: in a bidirectional reasoning system, "A cycle would be created where any slight evidence in favor of A would be amplified via B and fed back to A, quickly turning into a stronger conformation (of A and B), with no apparent factual justification."

What NAL1 does for the problem is to record only a *constant* part of the related experience for each judgment, and use it to determine whether two judgments are based on correlated evidence. [7] Though not perfect, it is a reasonable solution when resources are insufficient, and "reasonable solutions" are exactly what we expect from a non-axiomatic system. It is also similar to the strategy of human mind, since we usually have impressions about where our judgments come from, but such impressions are far from complete and accurate.

4.3.   Choice

What NAL1 should do when two conflicting judgments are based correlated evidence? Ideally, we would like to exactly record the contribution of each input judgment, and to subtract the weight of the overlapping section from the truth value of the conclusion. Unfortunately, this is impossible, because the experience recorded for each judgment is incomplete, as discussed previously.

The rule that NAL1 uses for the problem is a simple one: to take the judgment with a *higher confidence* (no matter what its *frequency* is) as the conclusion. For an adaptive system, if it must make a choice between

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[7] In a recent implementation, a "postmark" mechanism is used for this purpose, and it works well. See [40] for details.

conflicting judgments, the one related to much experience has a higher priority.

Choices are necessary in another situation: among competing answers. For a *selection* question with the form "$S \subset$?" ("? $\subset P$" can be similarly processed), the system is asked to find a term T (not S itself, of course) as a "typical element" in the intension of S. Ideally, the best answer is provided by a judgment "$S \subset T <1,\ 1>$" (here the truth value is represented by the *frequency* and *confidence* of the judgment). However, it is impossible, because *confidence* cannot reach 1 in NAL1. Therefore, we have to settle down with an answer which is the best the system can find under the constraints of available knowledge and resources.

Assuming the competing answers are

$$S \subset T_1\ <f_1,\ c_1>$$

$$S \subset T_2\ <f_2,\ c_2>\ .$$

Which one is better? Let us consider some special cases first:

1. $c_1 = c_2$, that is, the two answers are supported by the same amount of evidences (for example, both comes from statistical data of 100 samples). Obviously, the answer with a *higher frequency* is preferred, since that inheritance relation has more positive evidence than the other.
2. $f_1 = f_2 = 1$, that is, all available evidence is positive. Now the answer with a *higher confidence* is preferred, since it is more strongly confirmed by the experience.
3. $f_1 = f_2 = 0$, that is, all available evidence is negative. Now the answer with a *lower confidence* is preferred, since it is less strongly refuted by the experience.

From the special cases we can see that to set up a *general rule* to compare competing judgments in terms of which is more hopeful to be confirmed by the future experience of the system, we need to somehow combine the two numbers in a truth value into a single measurement.

In NAL1, *expectation*, e, is defined for this purpose. Different from a truth value (which is used to record past experience), an expectation (of a judgment) is used to predict future experience. "$e = 1$" means the system is absolutely sure that the inheritance relation under consideration will always be confirmed by future experience, "$e = 0$" for always refuted, and "$e = 0.5$" for no preference between a positive predication and a negative one. Intuitively, similar to *subjective probability* [21], e can be interpreted as the estimation about a future "inheritance frequency", or a bet the system will accept about a future "inheritance test". Under the assumption of insufficient knowledge, in NAL1 e only takes values in $(0,\ 1)$, with 0 and 1

as limits. For a selection question, the system takes the answer that has the higher expectation between the competing two.

To calculate e from $< f, c >$, we can see that under the assumption that the system make predictions according to its (past) experience, it is natural to use f as e's "first-order approximation". However, such a *maximum-likelihood estimate* is not good enough when c is small [16]. For example, if a hypothesis has been tested only once, nobody will take an *expectation* as 1 (if the test is a success) or 0 (if the test leads is failure).

Intuitively, e should be more "conservative" (more close to the "no preference" point, 0.5) than f under the consideration that the future may be different from the past. Here is where the confidence c affects e — the more evidence the system has accumulated, the more confident the system is (indicated by a larger c), then the more closely its predicted frequency e will be bound to its experienced frequency f.

Therefore, it is natural to define

$$e = c(f - 0.5) + 0.5.$$

Identically, it can be written as $c = (e - 0.5)/(f - 0.5)$ (when $f \neq 0.5$), so it says that c indicates the ratio that f is (to use Good's term in [16]) "squashed" to the "no preference point" to become e. When $c = 1$ (total evidence), $e = f$; when $c = 0$ (null evidence), $e = 0.5$.

To express the definition of e in the other two forms of truth value leads to interesting results.

When the truth value is represented as an interval, from the mappings among different forms of truth value that listed in Table 3, we get

$$e = 0.5(l + u)$$

It's exactly the *expectation of the future frequency*, that is, the middle point of the interval in which the frequency will be in the near future.

When the truth value is represented as weights of evidence, from the mappings we get

$$e = \frac{w^+ + \frac{k}{2}}{w + k}$$

which is a continuum with k as a parameter. This formula turns out to be closely related to Hardy's *beta-form* based continuum (with equally weighted positive evidence and negative evidence) [16], and Carnap's "λ-continuum" (with the *logical factor*, or the prior probability, to be $1/2$) [9]. Though interpreted differently, the three continua share the same formula, and make identical predictions. All the three continua have *Laplace's low of succession* as a special case (when $k = 2$), where the probability of a next success is estimated by $\frac{w^+ + 1}{w + 2}$.

Now we can see how the choice of the constant k can influence the behavior of a system. Let's compare a system A_1 (with $k = 1$) and a system A_2 (with $k = 10$). The problem is to make a choice between two competing answers "$S \subset P_1 \{w_1^+, w_1\}$" and "$S \subset P_2 \{w_2^+, w_2\}$" (where the truth values are represented as weights of evidence). It is easy to see that when $w_1 = w_2$ or $w_1^+/w_1 = w_2^+/w_2$, the preferences of the two systems have no difference. It is only when a system need to make a choice between a higher f and a higher c that will the k matter. For example, when $w_1^+ = w_1 = 2, w_2^+ = 5$, and $w_2 = 6$, A_1 will choose the first answer (since all of its evidence is positive), while A_2 will choose the second answer (since it is "better tested", and its frequency is not much lower than the other).

Therefore, k is one of the "personality parameters" of a reasoning system, in the sense that it indicates certain systematical preference or bias, and there is no "optimal value" for it in general. The larger k is, the more "conservative" the system is, in the sense that given the same amount of evidence, it always make less change in e, compared with a system with a smaller k. This parameter was called "flattening constant" by Good (see [16], where he also tried to estimate its value), and interpreted by him as a way to choose a prior probability distribution. The same parameter is interpreted by Carnap as "the relative weight" of the logical factor [9].

One reasonable alternative of the choice rule is to choose the conclusion probabilistically. The judgment with a higher confidence (for evaluation) or a higher expectation (for selection) is not always chosen as the answer, but is given a higher *probability* to be chosen. In this way, the decisions are more variable and indeterministic, so have some advantages in certain circumstances [18].

4.4. Syllogisms

The major inference rules in NAL1 are the (extended) syllogisms. When two judgments share a common term, they can be used as premises to infer the inheritance relations between the other two (unshared) terms.

Totally, there are four possible combinations of premises and conclusions, corresponding to the four figures of Aristotle's syllogisms:

1. From "$M \subset P < f_1, c_1 >$" and "$S \subset M < f_2, c_2 >$" to get "$S \subset P < f, c >$". This is Aristotle's *first figure*, and what Peirce called *deduction* [1, 30]. Let us refer to the function that calculate f and c from f_1, c_1, f_2, and c_2 as F_1.
2. From "$P \subset M < f_1, c_1 >$" and "$S \subset M < f_2, c_2 >$" to get "$S \subset P < f, c >$". This is Aristotle's *second figure*, and what Peirce called *abduction* (or *hypothesis*) [1, 30]. Let us refer to the function that calculate f and c from f_1, c_1, f_2, and c_2 as F_2.
3. From "$M \subset P < f_1, c_1 >$" and "$M \subset S < f_2, c_2 >$" to get "$S \subset P < f, c >$". This is Aristotle's *third figure*, and what Peirce called

induction [1, 30]. Let us refer to the function that calculate f and c from f_1, c_1, f_2, and c_2 as F_3.

4. From "$M \subset P < f_1, c_1 >$" and "$S \subset M < f_2, c_2 >$" to get "$P \subset S < f, c >$". This rule, not discussed by Aristotle and Peirce, was called the *fourth figure* by Aristotle's successors [4]. Let us refer to the function that calculate f and c from f_1, c_1, f_2, and c_2 as F_4.

Considering the conclusions that can be got by exchange the order of the premises, we get a complete syllogisms table for NAL1, Table 4, where F_i' is the function got by exchange $< f_1, c_1 >$ and $< f_2, c_2 >$ in the function F_i, (t_i) is the truth value of the premise J_i, and (F_i) is the truth value of the conclusion, calculated by F_i.

By extending the truth value and interpreting them both extensionally and intensionally, the syllogisms in NAL1 are quite different from Aristotle's and Peirce's, though still related to them.

The functions in Table 4 can be built by considering the relations among the involved truth values in terms of Triangular norm (T-norm) and Triangular conorm (T-conorm). T-norm and T-conorm are function from $[0, 1] \times [0, 1]$ to $[0, 1]$ that are monotonic, commutative, associative, and with boundary conditions satisfying the truth tables of the logical operators AND and OR, respectively [5, 6, 11]. They also can be extended to take more than two arguments.

The usage of T-norm and T-conorm in NAL1 is different from their usual usage [6, 11] in which they are used to determine the degree of certainty of the *conjunction* and *disjunction* of two propositions, respectively. In NAL1, T-norm $y = T(x_1, \ldots, x_n)$ is used when a quantity y is *conjunctively* determined by two or more other quantities $x_1, \ldots, x_n$, that is, $y = 1$ if and only if $x_1 = \cdots = x_n = 1$, and $y = 0$ if and only if $x_1 = 0$ or $\ldots$ or $x_n = 0$; T-conorm $y = S(x_1, \ldots, x_n)$ is used when a quantity y is *disjunctively* determined by two or more other quantities $x_1, \ldots, x_n$, that is, $y = 1$ if and only if $x_1 = 1$ or $\ldots$ or $x_n = 1$, and $y = 0$ if and only if $x_0 = \cdots = x_n = 0$. These quantities are not about the *conjunction* or *disjunction* of two judgments. [8]

Since the two premises are about two different inheritance relations, and the frequency and confidence of a judgment are determined by different factors, f_1, c_1, f_2, and c_2 can be referred to as mutually *independent* to each other in the sense that given any three of them, the last one cannot be determined, or even bounded approximately.

Given the ratio interpretations of the truth value and the independence among f_1, c_1, f_2, and c_2, it is natural For NAL1 to use the "probabilistic"

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[8] In NAL1, the conjunction or disjunction of two judgments is not defined as a judgment.

operators (see the comparison of different T-norms and T-conorms in [6]):

$$T(a, b) = ab, \quad S(a, b) = a + b - ab.$$

T-norm and T-conorm with more than two arguments are defined as:

$$T(x_1, \ldots, x_n) = T(T(x_1, \ldots, x_{n-1}), x_n),$$

$$S(x_1, \ldots, x_n) = S(S(x_1, \ldots, x_{n-1}), x_n).$$

From the point of view of NAL1, "deduction" is the extended "rule of transitivity". Only positive evidence for both premises is counted as positive evidence of the conclusion, and positive evidence for either of the premises is counted as relevant evidence of the conclusion. It follows that evidence that is positive to one premiss and negative to the other is negative evidence of the conclusion, and negative evidence to both premises is regarded as irrelevant to the conclusion. Consequently, we have the following boundary conditions:

$$f = \begin{cases} 1 & \text{if } f_1 = 1, \text{ and } f_2 = 1 \\ 0 & \text{if } f_1 = 1, \text{ and } f_2 = 0 \\ 0 & \text{if } f_1 = 0, \text{ and } f_2 = 1 \\ \text{undetermined} & \text{if } f_1 = 0, \text{ and } f_2 = 0 \end{cases}$$

The confidence of the conclusion is determined conjunctively by the amount of relevant evidence and the confidences of the premises, that means, it is determined by c_1, c_2, and $S(f_1, f_2)$, which measures the extent that "at least one premiss is positive". Obviously, c is 0 when any of the three factors is 0: either one premiss is supported by null evidence, or the premises are irrelevant to the conclusion. c is 1 only when all the three factors are 1, though in NAL1 confidences cannot reach 1, but have it as a limit.

In summary, we get

$$F_1: \quad \begin{aligned} f &= \frac{T(f_1, f_2)}{S(f_1, f_2)} &= \frac{f_1 f_2}{f_1 + f_2 - f_1 f_2} \\ c &= T(S(f_1, f_2), c_1, c_2) &= (f_1 + f_2 - f_1 f_2)\, c_1 c_2 \end{aligned}$$

In NAL1, "abduction" is the inference that from a shared element M of *intensions* of S and P to determine the truth value of "$S \subset P$", and "induction" is the inference that from a shared element M of *extensions* of S and P to determine the truth value of "$S \subset P$". From the symmetry between extension and intension, we know that $F_2 = F_3'$ (and $F_3 = F_2'$). Therefore, we only need to discuss one of them, say F_3.

In determining the truth value of "$S \subset P$" from the common instance M of the two terms, the truth values of the two premises have different

functions. The frequency of "$M \subset P$", f_1, estimates the frequency of the conclusion, since we are taking the property ("having P as part of intension") of a special term M as a property of the general term S. On the other hand, f_2, c_1 and c_2 *conjunctively* determines to what extent M can be counted as a piece of relevant evidence of the conclusion. At most, we can only get *one* term (indicated by $w = 1$, according to the interpretation of the truth value) in the extension of S as evidence. Therefore, for the conclusion, we take $w = T(f_2, c_1, c_2)$. Writing as functions from f_1, c_1, f_2, c_2 to f and c, we have

$$
\begin{aligned}
F_2 : \quad f &= f_2 \\
c &= \frac{f_1 c_1 c_2}{f_1 c_1 c_2 + k} \\
F_3 : \quad f &= f_1 \\
c &= \frac{f_2 c_1 c_2}{f_2 c_1 c_2 + k}
\end{aligned}
$$

Defined as above, abduction and induction are no longer "inversed deductions" [25, 30], and the difference between them and deduction is still there: deductive conclusions are usually much more confident (with 1 as upper bound) than abductive and inductive conclusions (with $\frac{1}{1+k}$ as upper bound). [9]

Using F_2 or F_3, we can define NAL1's conversion rule. In term logics, "conversion" is a inference from a single premiss to a conclusion by interchanging the subject term and the predicate term [4].

Now we can see it as a special case of abduction by taking "$P \subset S < f_0, c_0 >$" and "$S \subset S < 1, 1 >$" (a tautology) as premises. As a result, we get the truth value function of the conversion rule:

$$
\begin{aligned}
f &= 1 \\
c &= \frac{f_0 c_0}{f_0 c_0 + k}
\end{aligned}
$$

Similarly, we can get the same result by seeing conversion as a special case of induction with "$P \subset P < 1, 1 >$" and "$P \subset S < f_0, c_0 >$" as premises.

Now how to understand the conversion rule directly? From our interpretation of the truth value, we see that the positive evidence for "$S \subset P$" are also positive evidence for "$P \subset S$", but the negative evidence of the former is irrelevant to the latter. It means that the conclusion can only be conformed, but never refuted, by such a rule (this is different from the cases in ETL and ITL). As a result, we have $f = 1$ in all situations. For the weight of the conclusion, we know that it is at most 1, and it only happens

---

[9] Here we can see the personality parameter k's another function: to indicate the relative confidence of abductive/inductive conclusions. Comparatively speaking, a system with a small k relies more on abduction and induction, while a system with a large k relies more on deduction.

when $f_0 = c_0 = 1$. In that case, P is in the extensions of both S and P. Therefore, we take $w = T(f_0, c_0)$.

This analysis lead us to the truth value function of the "fourth figure", where we have

$$F_4 : \quad \begin{aligned} f &= 1 \\ c &= \frac{f_1 f_2 c_1 c_2}{f_1 f_2 c_1 c_2 + k} \end{aligned}$$

That is, just like the situation of the conversion rule, no negative evidence for the conclusion can be collected in this way, and w of the conclusion is determined conjunctively by f_1, f_2, c_1, and c_2. Only when $f_1 f_2 c_1 c_2 = 1$, can the conclusion get a support measured by $w = 1$, since then we have "$P \sqsubset S$", that is, P is in the extensions of both S and P.

There is another interesting result. From "$M \subset P < f_1, c_1 >$" and "$M \subset S < f_2, c_2 >$", NAL1 can directly get "$S \subset P < f_1, \frac{f_2 c_1 c_2}{f_2 c_1 c_2 + k} >$" by induction. However, there is also an indirect way to get a truth value for "$S \subset P$" from the same premises: first, by conversion, the second premiss derives "$S \subset M < 1, \frac{f_2 c_2}{f_2 c_2 + k} >$", then, from this judgment and the first premiss, NAL1 can get a deductive conclusion "$S \subset P < f_1, \frac{f_2 c_1 c_2}{f_2 c_2 + k} >$". Compared with the direct result, the indirect conclusion has the same frequency, but a lower confidence.

Similarly, abduction and the fourth-figure can be replaced by conversion-then-deduction, but with a confidence loss. On one hand, the results show that all inference rules in NAL1 will cause certain information loss (though having some other information preserved), therefore direct conclusions are more confident. On the other hand, the fact that the same frequency is arrived by following different inference paths shows that the truth value functions are not coined individually in *ad hoc* ways, but closely related to each other, since all of them are based on the same semantic interpretation of the truth value.

4.5. A Summary

Similar to IL, ETL and ITL, the memory of NAL1 can also be described as a network, where "$\subset$" is the only type of link, with truth value as weight, between nodes (terms). To solve an evaluation question means to determine the weight of a link, given its beginning and ending node; to solve a selection question means to locate a node with the strongest link (that is, the highest expectation) from or to a given node. Since by applying rules, both the topological structure of the network and the values of the weights can be changed, what the system does is much more than searching a static network for the desired link or node.

The choice rules are for choosing between competing links; the revision rule is for merging two links that share both the start and end nodes but supported by distinct bodies of evidence; the conversion rule is for reversing

a link; and finally, the syllogisms are for chaining two adjacent links into a new one, where different combinations of directions correspond to different types of inference. We can represent them by the patterns in Figure 2, where a link (from subject term to predicate term) with a single arrow is a premiss, and a link with a double arrow is a conclusion (the symmetric conclusions in the syllogisms are omitted). This "network interpretation" of NAL1 reminds us Minsky's comment [26]:

> For the purposes of psychology, we'd better to set aside the dubious ideal of faultless deduction and try, instead, to understand how people actually deal with what is *usual* or *typical*. To do this, we often think in terms of causes, similarities, and dependencies. What do all these forms of thinking share? They all use different ways to make chains.

We also collect all the functions, in all the three forms of truth value, in Table 5.

It is possible to find direct intuitive justifications for a function in a form that is different from the previously discussed ones (for example, Bai Shuo in [2] also reached the revision rule in the ratio form from a different starting point), but such justifications are not always obvious.

5. An Example

Up to now, we have completely defined a non-axiomatic logic, NAL1, with its grammar, semantics, and inference rules.

To get a non-axiomatic reasoning system, we need to provide a memory and a control mechanism, which are adaptive and work with insufficient knowledge and resources. A description of the two components can be found in [40].

In this section, let us see how NAL1 works on an example.

5.1. Background Knowledge

The experience of the system, provided by a human user, consists of the following judgments:

(J_1) $birds \subset flyers$ $<1, 0.8>$
(J_2) $doves \subset birds$ $<1, 0.8>$
(J_3) $doves \subset swimmers$ $<0, 0.8>$
(J_4) $swans \subset birds$ $<1, 0.8>$
(J_5) $swans \subset flyers$ $<1, 0.8>$
(J_6) $swans \subset swimmers$ $<1, 0.8>$

(J_7) *penguins* $\subset$ *birds* $<$1, 0.8$>$
(J_8) *penguins* $\subset$ *flyers* $<$0, 0.8$>$
(J_9) *penguins* $\subset$ *swimmers* $<$1, 0.8$>$

The truth values are represented in the "$< frequency,\ confidence >$" form. To simplify the presentation, only two frequencies appear in the input judgments: 1 and 0, corresponding to "all evidence is positive" and "all evidence is negative", respectively. All the input judgments are assigned a confidence value 0.8 by the user. Under the presumption that $k = 1$ (the constant for "near future" defined previously), it means that all the judgments are supported by evidence with $w = 4$ (so $c = \frac{w}{w+k} = 0.8$).

5.2. Deduction

The first question is: "*doves @flyers*", that is, "Are doves flyers?".
From J_1 and J_2, by deduction, the system get:

(J_{10}) *doves* $\subset$ *flyers* $<$1, 0.64$>$

That means, because all available evidence is positive, the answer is "Yes". However, the confidence of the conclusion is lower than either of the premises', so it is easier to be revised by future evidence. Generally speaking, all syllogistic inferences cause confidence loss, that is, the confidence of the conclusion is always lower than lowest confidence of the premises.

5.3. Induction

The second question is: "*birds @swimmers*", that is, "Are birds swimmers?".
From J_4 and J_6, by induction, the system get:

(J_{11}) *birds* $\subset$ *swimmers* $<$1, 0.39$>$

The conclusion is "Yes", but not confident, compared with the above deductive conclusion. If the time supply is very tight, the system will report the answer to the user, then turn to other tasks. Otherwise, from J_7 and J_9, the system can get another inductive conclusion:

(J_{12}) *birds* $\subset$ *swimmers* $<$1, 0.39$>$

J_{11} and J_{12} looks identical, but since they are supported by distinct bodies of evidence, the system can use the revision rule to get a summarized conclusion:

(J_{13}) *birds* $\subset$ *swimmers* $<$1, 0.56$>$

Though the *frequency* remains unchanged, the *confidence* increases, reflecting the accumulation of evidence.

If the system continue to work on this question, negative evidence will be found by an induction from J_2 and J_3:

(J_{14}) $birds \subset swimmers$ $<0,\ 0.39>$

Obviously, it conflicts with J_{13}. Using the revision rule once again, the system get:

(J_{15}) $birds \subset swimmers$ $<0.66,\ 0.66>$

That means "Most birds are swimmers." Generally speaking, revision cause a compromise between the frequencies of the premises (weighted by a function of the corresponding confidence), and a monotonically increase of confidence.

Another property of NARS that can be noticed here is: which answers are possible are determined by the "logic components" of the system, such as the inference rules, but which possible answer is actually reported in a certain situation is determined by the "control components" of the system, such as the memory structure and resources supply. In this example, J_{11}, J_{12}, J_{13}, J_{14}, and J_{15} are all possible answers. If they compete with each other, J_{15} will win, because it has the highest confidence. [10]

5.4. Abduction

The third question is: "$? \subset birds$", that is, to look for a typical type of bird.

At the first glimpse, we can see that J_2, J_4, and J_7 are equally good answers for this question. However, if the system can spend more resources on this question, even the input judgments can be revised by taking other evidence into account.

From J_1 and J_5, by abduction, the system get:

(J_{16}) $swans \subset birds$ $<1,\ 0.39>$

That is, $swans$ has a property of $birds$, that is, being $flyers$, therefore "$swans \subset birds$" get some positive evidence. Because J_{16} and J_4 are based on different experience sections of the system, they can be combined by the revision rule to get

(J_{17}) $swans \subset birds$ $<1,\ 0.82>$

Now the system's belief about "Swans are birds" become stabler, as the result of considering more evidence.

Similarly, from J_1 and J_8, the system gets another abductive conclusion:

(J_{18}) $penguins \subset birds$ $<0,\ 0.39>$

Because $penguins$ lack the property of "being $flyers$", they are not $birds$, in this aspect. When this judgment is used to revise J_7, the system get:

---

[10] The system will not use J_{14} and J_{15} for another revision, because they are based on correlated evidence.

$(J_{19})\ penguins \subset birds < 0.86,\ 0.82 >$

Now, after summarizing evidence, the system still believe *penguins* are *birds*, but atypical ones. Because the negative evidence is found by comparing the properties (intensions) of *penguins* and *birds*, the conclusion cannot be interpreted extensionally as "86% penguins are birds". Now the frequency value, 0.86, is more similar to the "membership grade" in fuzzy set theory or "degree of representativeness" studied in cognitive psychology.

For *doves*, the system can do similar inferences. However, since "Doves are flyers" is not an input judgment, but a derived one, and in the derivation "Doves are birds" has been used, the corresponding abductive conclusion cannot be used to revise "Doves are birds", because the evidence of the two judgments are correlated. We can see that the problem caused by "bidirectional inferences" (discussed in subsection 3.2) does not appear here.

As a result, *swans* will be referred to by the system as a better instance of *birds*, compared with *penguins* and *doves*.

6. Conclusions

The major aim of this paper is to completely define NAL1. The detailed comparisons of it with other theories are left for other papers. The most important contribution of NAL1 is that it is designed to work in a reasoning system which is adaptive, finite, real-time, and open. To do this, ideas including term-oriented language, experience-grounded semantics, local syllogistic rules, and so on, are invented and applied.

The most distinguishing features of NAL1 is its ability of uniformly representing and processing multiple types of uncertainty (including randomness, fuzziness, and ignorance), and doing multiple types of inference (including deduction, induction, abduction, and revision).

The major limitation of NAL1 is its expressive capacity. With a single inheritance relation and atomic terms, NAL1 has problems in representing many types of knowledge. However, this limitation belongs to NAL1 only, rather than to the NARS project as a whole. Actually, as the simplest one in the NAL family, NAL1 is deliberately equipped with a formal language that is as simple as possible. It is not designed to be used directly for practical purpose, but to be a prototype by which some ideas can be demonstrated and tested. In its future extensions, other inheritance relations (such as "$\in$" and "$=$") and structured terms (such as the unions, intersections, differences, and ordinary relations of terms) will be introduced, and hopefully the system will become competent in its expressive capacity.

NAL1 is not a "logicist" approach, since it does not use first order predicate logic and model-theoretic semantics, and have non-deductive rules. It is still a logic, however, in the sense that it uses a domain-independent formal language to represent knowledge, and uses formal rules to capture patterns appearing in human reasoning [37]. By naming it a "Non-Axiomatic Logic", I am trying to show that, from the viewpoint of artificial intelligence, the problems of the traditional "symbolic AI" [3, 25, 34] are not caused by the ideas like "formalization", "symbolization", "logical inferences", and so on, but by the ideas like "axiomatization", "computation", "binary logics", "consistent and complete system", and other concepts that explicitly or implicitly assume the sufficiency of knowledge and resources. Currently, the two classes of ideas are not clearly distinguished. I believe the second class is improper for AI, but the first class can still be fruitful.

In NAL1, all judgments are about "to what extent one term can be used as another", and all inferences are about the "can-be-used-as" relation (formally defined as the inheritance relation), which remind us mathematician Ulam's comment on logic and artificial intelligence: "It is the word 'as' that must be mathematically formalized ... Until you do that, you will not get very far with your AI problem". Hofstadter discussed this opinion in [17], and generalized it into "Ulam's thesis": "AS is the key to AI". NAL1 can be seen as a primary step in this direction.

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	$\{w^+,\, w\}$	$<f,\, c>$	$[l,\, u]$
$\{w^+,\, w\}$		$w^+ = k\,\frac{fc}{1-c}$ $w = k\,\frac{c}{1-c}$	$w^+ = k\,\frac{l}{u-l}$ $w = k\,\frac{1-(u-l)}{u-l}$
$<f,\, c>$	$f = \frac{w^+}{w}$ $c = \frac{w}{w+k}$		$f = \frac{l}{1-(u-l)}$ $c = 1 - (u-l)$
$[l,\, u]$	$l = \frac{w^+}{w+k}$ $u = \frac{w^++k}{w+k}$	$l = fc$ $u = 1 - c(1-f)$	

Table 3. Relations among uncertainty measurements

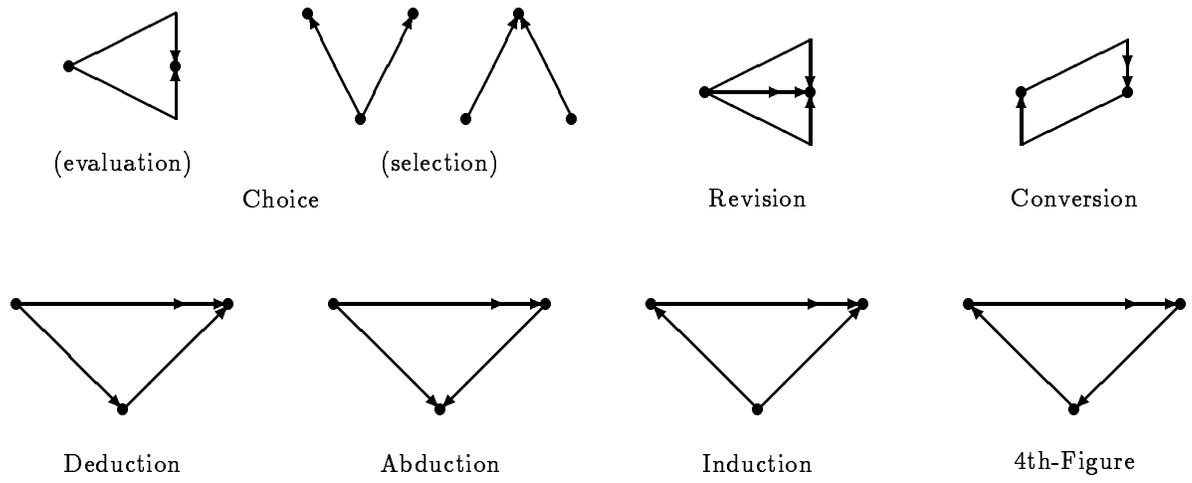| $J_2 \setminus J_1$	$M \subset P(t_1)$	$P \subset M(t_1)$
$S \subset M(t_2)$	$S \subset P(F_1)$ $P \subset S(F_4')$	$S \subset P(F_2)$ $P \subset S(F_2')$
$M \subset S(t_2)$	$S \subset P(F_3)$ $P \subset S(F_3')$	$S \subset P(F_4)$ $P \subset S(F_1')$

Table 4. Inference rules of NAL1

Figure 2. Operations on links

	$\{w^+, w\}$(and w^-)	$<f, c>$ (and $i = 1 - c$)	$[l, u]$(and $i = u - l$)
Revision	$w^+ = w_1^+ + w_2^+$ $w = w_1 + w_2$	$f = \dfrac{\frac{c_1}{i_1}f_1+\frac{c_2}{i_2}f_2}{\frac{c_1}{i_1}+\frac{c_2}{i_2}}$ $c = \dfrac{\frac{c_1}{i_1}+\frac{c_2}{i_2}}{\frac{c_1}{i_1}+\frac{c_2}{i_2}+1}$	$l = \dfrac{\frac{l_1}{i_1}+\frac{l_2}{i_2}}{\frac{1}{i_1}+\frac{1}{i_2}-1}$ $u = \dfrac{\frac{l_1}{i_1}+\frac{l_2}{i_2}+1}{\frac{1}{i_1}+\frac{1}{i_2}-1}$
Conversion	$w^+ = \dfrac{w_0^+}{w_0+k}$ $w = \dfrac{w_0^+}{w_0+k}$	$f = 1$ $c = \dfrac{f_0 c_0}{f_0 c_0+k}$	$l = \dfrac{l_0}{l_0+k}$ $u = 1$
Deduction	$w^+ = \dfrac{w_1^+ w_2^+}{w_1+w_2+k-\frac{w_1^- w_2^-}{k}}$ $w = \dfrac{w_1^+ w_2 + w_1 w_2^+ - w_1^+ w_2^+}{w_1+w_2+k-\frac{w_1^- w_2^-}{k}}$	$f = \dfrac{f_1 f_2}{f_1+f_2-f_1 f_2}$ $c = (f_1 + f_2 - f_1 f_2)\, c_1 c_2$	$l = l_1 l_2$ $u = 1 - l_1 - l_2 + l_1 u_2 + l_2 u_1$
Abduction	$w^+ = \dfrac{w_1^+ w_2^+}{(w_1+k)(w_2+k)}$ $w = \dfrac{w_1^+ w_2}{(w_1+k)(w_2+k)}$	$f = f_2$ $c = \dfrac{f_1 c_1 c_2}{f_1 c_1 c_2+k}$	$l = \dfrac{l_1 l_2}{l_1(1-i_2)+k}$ $u = \dfrac{l_1 l_2+k}{l_1(1-i_2)+k}$
Induction	$w^+ = \dfrac{w_1^+ w_2^+}{(w_1+k)(w_2+k)}$ $w = \dfrac{w_1 w_2^+}{(w_1+k)(w_2+k)}$	$f = f_1$ $c = \dfrac{f_2 c_1 c_2}{f_2 c_1 c_2+k}$	$l = \dfrac{l_1 l_2}{l_2(1-i_1)+k}$ $u = \dfrac{l_1 l_2+k}{l_2(1-i_1)+k}$
4th-Figure	$w^+ = \dfrac{w_1^+ w_2^+}{(w_1+k)(w_2+k)}$ $w = \dfrac{w_1^+ w_2^+}{(w_1+k)(w_2+k)}$	$f = 1$ $c = \dfrac{f_1 f_2 c_1 c_2}{f_1 f_2 c_1 c_2+k}$	$l = \dfrac{l_1 l_2}{l_1 l_2+k}$ $u = 1$

Table 5. Truth value functions