3D Object Digitization: Volume and Surface Area Estimation

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Abstract

Measuring volume and surface area of objects given its digitizations are important problems in 3D image analysis. Good estimators should be multigrid convergent, i.e. the error goes to zero with increasing sampling density. We will give such estimators both for volume and for surface area estimation based on simple counting of voxels.

1 Introduction

A fundamental task of knowledge representation and processing is to infer properties of real objects or situations given their representations. In spatial knowledge representation and, in particular, in computer vision and medical imaging, real objects are represented in a pictorial way as finite and discrete sets of pixels or voxels. The discrete sets result by a quantization process, which is naturally realized by technical devices like Computer Tomography scanners, CCD cameras or document scanners. A fundamental question addressed in spatial knowledge representation is: How can we measure certain properties of an object by looking at its discrete representation? Two of the most important object properties are volume and surface area. While there exist a lot of approaches for precise volume measurement, surface area estimation seems to be not as simple: With increasing sampling density most known algorithms do not necessarily converge to the original surface area. We analyse the problem of multigrid convergent surface area estimation and suggest that one should use *semi-local* algorithms, since local algorithms do not seem to be multigrid-convergent and there exists no proof for any global algorithm. We give an example of a semi-local surface area estimator and prove that it is multigrid-convergent.

2 Preliminaries

Let A be any subset of $\mathbb{R}^3$. The *complement* of A is denoted by A^c. All points in A are *foreground* while the points in A^c are called *background*. The *open ball* in $\mathbb{R}^3$ of radius r and center c is the set $\mathcal{B}^0_r(c) = \{x \in \mathbb{R}^3 \mid d(x,c) < r\}$, and the *closed ball* in $\mathbb{R}^3$ of radius r and center c is the set $\overline{\mathcal{B}}_r(c) = \{x \in \mathbb{R}^3 \mid d(x,c) \leq r\}$. Whenever $c = (0,0,0) \in \mathbb{R}^3$, we write $\mathcal{B}^0_r$ and $\overline{\mathcal{B}}_r$. The *r-dilation* $A \oplus \mathcal{B}^0_r$ of a set A is the union of all open r-balls with center in A. We say that an open ball $\mathcal{B}^0_r(c)$ is *tangent* to ∂A at a point $x \in \partial A$ if $\partial A \cap \partial B^0_r(c) = \{x\}$. We say that an open ball $\mathcal{B}^0_r(c)$ is an *osculating open ball of radius r to ∂A at point $x \in \partial A$* if $\mathcal{B}^0_r(c)$ is tangent to ∂A at x and either $\mathcal{B}^0_r(c) \subseteq A^0$ or $\mathcal{B}^0_r(c) \subseteq (A^c)^0$. Now, we define r-regular sets in $\mathbb{R}^3$ (refer to Fig. 1):

Definition 1 *A set $A \subset \mathbb{R}^3$ is called* r-regular *if, for each point $x \in \partial A$, there exist two osculating open balls of radius r to ∂A at x such that one lies entirely in A and the other lies entirely in A^c.* ◇

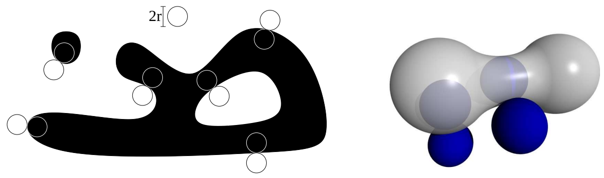


Figure 1. For each boundary point of a 2D/3D r-regular set there exists an outside and an inside osculating open disc/ball of radius r.

Any set S which is a translated and rotated version of the set $\frac{2 \cdot r'}{\sqrt{3}}\mathbb{Z}^3$ is called a *cubic r'-grid* and its elements are called *sampling points*. Note that the distance $d(x,p)$ from each point $x \in \mathbb{R}^3$ to the nearest sampling point $s \in S$ is at most r'. The *voxel* $\mathcal{V}_S(s)$ of a sampling point $s \in S$ is its *Voronoi region* $\mathbb{R}^3$: $\mathcal{V}_S(s) = \{x \in \mathbb{R}^3 \mid d(x,s) \leq d(x,q),\ \forall q \in S\}$, i.e., $\mathcal{V}_S(s)$ is the set of all points of $\mathbb{R}^3$ which are at least as close to s as to any other point in S. In particular, note that $\mathcal{V}_S(s)$ is a cube whose vertices lie on a sphere of radius r' and center s. The following method for reconstructing the object from the set of included sampling points is the 3D generalization of the 2D *Gauss digitization* (see [1]) which has been used by Gauss to compute the area of discs in 2D:

Definition 2 *Let S be a cubic r'-grid, and let A be any subset of $\mathbb{R}^3$. The union of all voxels with sampling points lying in A is the* digital reconstruction of A with respect to S, $\hat{A} = \bigcup_{s \in (S \cap A)} \mathcal{V}_S(s)$. ◇

3 Volume and Surface Estimation

The estimation of object properties like volume and surface area given only a digitization is an important problem in image analysis. Here we will show that both can be computed with high accuracy if the original object is r-regular. In [2] and [3] we introduced several methods to reconstruct a sampled 3D object with only a small geometric and no topological error. This can directly be used to give absolute bounds for the difference between the reconstructed and the original volume:

Let A' be the digital reconstruction of an r-regular object A with a cubic r'-grid S with $2r' < r$. Without loss of generality let $S = \mathbb{Z}^3$ (A and S can always be transformed such that this is true). Now let $\{c_i\} = \mathbb{Z}^3 - (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ be the set of *corner points* of voxels centered in $s_i \in S$. Then each r'-ball $\mathcal{B}_{r'}(c_i)$ has exactly eight sampling points s_i on its surface. The voxels of these eight sampling points contain c_i as corner point. Now let $C \subset \{c_i\}$ be the set of corner points whose eight sampling points are not all inside or all outside the object A. Then due to r-regularity of A the union U of all r'-balls with centers in C supercovers the boundary ∂A (a proof can be found in [4]). Moreover U covers not only the surface of the digital reconstruction, but also the surface of any topology preserving reconstruction method being presented in [2] and [3], i.e. the surfaces of trilinear interpolation, marching cubes reconstruction, majority interpolation and sphere union. Thus the original set and all the different reconstruction methods differ only inside of U and since $\mathbb{V}(U) \leq n\pi r'^2$ with n being the number of points in C, the difference between the original volume and the volume of one of the reconstructions, i.e. the volume reconstruction error is at most $n\pi r'^2$. With $\lim_{r' \to 0} \mathbb{V}(U) = 0$ follows that this volume estimation method is multigrid convergent for any r-regular image.

Multigrid convergence of a function $f_{r'}$ on a digital representation of an object with sampling grid size r' means that $\lim_{r' \to 0} f_{r'}$ is equal to the value for the continuous object.

Surface estimation is not as simple as volume estimation. Kenmochi and Klette showed that local surface estimation methods are not multigrid convergent [5]. This is quite reasonable, since any local surface area estimation method (local means that the size of the area around a local cube which is used for approximating the surface locally is fixed relatively to the sampling grid size) based on binary images allows only a finite number of different surface patches, while even the number of different orientations of planar surfaces is infinite.

This means we need a non-local method in the sense that the size of the area around a local cube which is used for approximating the surface locally has to increase with increasing sampling density.

In the literature, two main approaches for global surface area estimation exist. While Klette et al. [6] use a digital plane segmentation process without giving a proof of multigrid convergence, Sloboda et al. [7] define a multigrid convergent method based on the relative convex hull of the discrete object, but no efficient algorithm exists to compute the relative convex hull.

The first and as far as we know up to now the only approach giving a multigrid convergent algorithm was introduced by Coeurjolly et al. [8]. They estimate the surface normals and use this to compute a surface area approximation. They prove that their algorithm is multigrid convergent if the size of the local area which is used to estimate a surface normal vector decreases with $O(\sqrt{\theta})$, where θ is a measure for the grid size. Thus their approach is local in the sense that the used area converges to zero relatively to the object size, but it is global in the sense that it converges to infinity relatively to the grid size, since $\lim_{\theta \to 0} \frac{O(\sqrt{\theta})}{\theta} = \infty$. We will call such methods *semi-local*. Note that in their experiments Coeurjolly et al. used a fixed minimal size for the used local area, such that their implementation is not multigrid convergent.

We think that using a semi-local approach for surface area estimation is the right choice. In this paper we will show that semi-local surface area estimation can be done in a much more simple way than proposed by Coeurjolly et al. by simply counting certain sampling points. While in [8] the estimation of surface normals was used to approximate the surface area, we will measure the volume of a thick representation of the surface. The idea is that with the thickness of this volume going to zero, the surface can be approximated by dividing the volume by the thickness. The volume can be estimated by counting voxels. Since the volume estimation has to converge faster than the size reduction of the surface, we have to increase the sampling density faster than decreasing the thickness of the surface representation. That is why our approach is semi-local. The basic property which makes our approach possible, the connection between surface area and volume, is given by the following lemma:

Lemma 3 *Let A be an r-regular object. Then the surface area $\mathbb{A}(\partial A)$ is equal to $\lim_{s\to 0}\frac{1}{2s}\mathbb{V}(\partial A\oplus\mathcal{B}_s)$, where $\partial A\oplus\mathcal{B}_s$ can be seen as a thick representation of the surface ∂A with thickness $2s$.*

Proof: Let $\{T_k\}$ be a polygonal surface approximation of ∂A such that each polygon T_k is a triangle such that the distance between any two of the three triangle points $t_{k,1}, t_{k,2}, t_{k,3} \in \partial A$ is bounded by s (This can be done by using the MMC algorithm introduced in [3]). Now let $n_{k,1}, n_{k,2}, n_{k,3}$ be the normal vectors of ∂A in $t_{k,1}, t_{k,2}, t_{k,3}$, and let V_k and W_k be the triangles which one gets by projecting T_k along the normals onto the two planes being parallel to the plane containing T_k with distance s. Further let P_k be the convex hull of the six corner points of V_k and W_k. Then P_k is a prismoid and its volume is $\mathbb{V}(P_k) = \frac{s}{3}(\mathbb{A}(V_k)+4\mathbb{A}(T_k)+\mathbb{A}(W_k))$. The union of the prismoids approximates $\mathbb{V}(\partial A\oplus\mathcal{B}(s))$, thus:

$$\sum_{k\in\mathbb{N}}\mathbb{V}(P_k) = \sum_{k\in\mathbb{N}}\left(\frac{s}{3}(\mathbb{A}(V_k)+4\mathbb{A}(T_k)+\mathbb{A}(W_k))\right)$$

$$= \frac{s}{3}(\sum_{k\in\mathbb{N}}\mathbb{A}(V_k)+4\sum_{k\in\mathbb{N}}\mathbb{A}(T_k)+\sum_{k\in\mathbb{N}}\mathbb{A}(W_k))$$

For $s\to 0$ the vectors of any triangle T_k become parallel and thus $\mathbb{A}(V_k)\to\mathbb{A}(T_k)$ and $\mathbb{A}(W_k)\to\mathbb{A}(T_k)$. This leads to

$$\lim_{s\to 0}\frac{1}{2s}\mathbb{V}(\partial A\oplus\mathcal{B}_s) = \lim_{s\to 0}\sum_{k\in\mathbb{N}}\frac{\mathbb{V}(P_k)}{2s}$$

$$= \lim_{s\to 0}\frac{1}{6}\left(\sum_{k\in\mathbb{N}}\mathbb{A}(V_k)+4\sum_{k\in\mathbb{N}}\mathbb{A}(T_k)+\sum_{k\in\mathbb{N}}\mathbb{A}(W_k)\right)$$

$$= \lim_{s\to 0}\frac{1}{6}\left(6\sum_{k\in\mathbb{N}}\mathbb{A}(T_k)\right) = \lim_{s\to 0}\sum_{k\in\mathbb{N}}\mathbb{A}(T_k) = \mathbb{A}(\partial A).$$

□

Now we can use the measurement of volumes for surface area estimation. In order to get a multigrid convergent method for surface estimation, we must measure the volume of a thick representation of the surface and we must guarantee that (1) the thickness parameter s converges to zero and (2) the estimation accuracy of its volume also converges to zero. This is possible by choosing $\lim_{r'\to 0} s = 0$ and $\lim_{r'\to 0}\frac{r'}{s} = 0$, i.e. r converges faster to zero than s. The last remaining problem is to estimate the volume of a thick representation of the surface by using only the information which sampling points are inside the object and which sampling points are outside. This is done as follows:

We know that the union U of all r'-balls with centers in C covers ∂A. Thus the $s+r'$-dilation of C covers $\partial A\oplus\mathcal{B}_s$, i.e. the thick representation of ∂A of thickness $2s$. Otherwise since $\partial A\oplus\mathcal{B}_{r'}\supset C$ for any r-regular set A with $r'<r$, we know that $\partial A\oplus\mathcal{B}_{(r'+(s-r'))}$ covers the $(s-r')$-dilation of C. Thus the volume of $\partial A\oplus\mathcal{B}_s$ can be approximated by counting the sampling points inside $C\oplus\mathcal{B}_s$ (see Fig. 2). With $N := \sharp\left\{s_i \,\middle|\, |s_i-c_j|\le s, c_j\in C\right\}$ follows for the volume of the thick representation:

$$\mathbb{V}(\partial A\oplus\mathcal{B}_s) = \lim_{r'\to 0}\frac{2}{\sqrt{3}}r'^3\cdot N.$$

Thus

$$\mathbb{A}(\partial A) = \lim_{s\to 0}\frac{1}{2s}\mathbb{V}(\partial A\oplus\mathcal{B}_s)$$

$$= \lim_{s\to 0,\,\frac{r'}{s}\to 0}\frac{1}{2s}\frac{2}{\sqrt{3}}r'^3\cdot N = \lim_{s\to 0,\,\frac{r'}{s}\to 0}\frac{r'^3}{\sqrt{3}s}\cdot N$$

Thus the output of the following algorithm converges to the true surface area:

(1) Let A be an r-regular set ; $n = 0$
(2) do
(3) $r' = \left(\frac{1}{2}\right)^n$; $s = \left(\frac{3}{4}\right)^n$.
(4) Compute the intersection of the sampling points s_i of the r'-grid $\frac{2}{\sqrt{3}}r'\cdot\mathbb{Z}^3$ with A.
(5) Compute the set C of center points c_j of the cubic neighborhood configurations C_j which consist of both foreground and background sampling points.
(6) Count the number N of sampling points s_i with distance smaller than s to some $c_j\in C$.
(7) $\mathbb{A}_n = \frac{r'^3}{\sqrt{3}s}\cdot N$; $n = n+1$
(8) loop until convergence of $\mathbb{A}_n$.
(9) return $\mathbb{A}_n$.

The presented method is local relatively to the regularity constraint r, i.e. relatively to the object size, but it is global relatively to the size of the sampling grid. That's why we call our approach *semi-local*. We think that the idea of a semi-local method is the best choice for dealing with the problem of surface area estimation, since local methods are not multigrid convergent and it seems to be difficult to prove the convergence of global methods. Our solution to the problem of multigrid convergent surface area estimation is extremely simple. In order to find the sampling points with distance smaller than s (step (6) of the algorithm), one can use a linear-time algorithm for Euclidean distance transform [9]. Then the above algorithm only needs linear time for a given sampling resolution relatively to the number of sampling points.

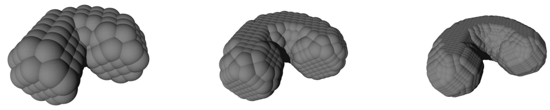
Although the class of r-regular objects is very general, a lot of objects of interest are not r-regular for any r. Nevertheless our algorithm is multigrid convergent if the surface of an object is almost everywhere differentiable, since then the percentage of the surface which behaves r-regular (i.e. there exists an outside ans an inside osculating r-ball) goes to 100% for $r \to 0$. Note that this is true for nearly any object of interest.

Theorem 4 *Let A be a continuous object with bounded curvature with except a set E that is a finite union of curves of finite length (sharp edges). Then the above surface area estimation algorithm converges to the true surface area $\mathbb{A}(\partial A)$, i.e., the mulitgrid convergence is true for A.*

Proof: Let $B_t = \partial A \setminus (E \oplus \mathcal{B}_t)$ be the surface of A without a t-neighborhood of E. Then B_t is a finite union of compact surface patches $A_1 \cup \cdots \cup A_n$. The patches are disjoint, and their curvature is bounded by some constant u. Taking $r = min(t, u)$, B_t is an r-regular surface, i.e., for every surface interior point $x \in B_t$, there exist two different r-balls that intersect B_t in exactly x. This implies the convergence of the above algorithm to $\mathbb{A}(B_t)$. If t goes to zero, the error due to the wrong surface area measurement inside $\partial A \cap (E \oplus B_t)$ converges to zero and the surface area of B_t goes to the surface area of A. Thus the algorithm converges to $\mathbb{A}(\partial A)$. □

4 Conclusions

We have showed that the concept of 3D r-regular sets is applicable for volume and surface area estimation in digital images. Several 3D object reconstruction methods which we introduced in two previous papers and which reconstruct the correct topology of the original object can directly be used for volume estimation. We discussed why surface area estimation is much more complicated and is not possible by using such local reconstruction methods. As an alternative we introduced a new surface area estimation algorithm and proved that it is multigrid convergent not only for r-regular objects but for nearly every object of interest. This algorithm is semi-local, i.e. local relatively to the object and global relatively to the size of the sampling grid. We think that the concept of semi-local surface area estimation is the key method for developing multigrid-convergent algorithms.



$n = 1$ $n = 2$ $n = 3$

Figure 2. $C \oplus \mathcal{B}_s$ **appoximates** ∂A **with increasing** n**.**

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