3D Object Digitization: Majority Interpolation and Marching Cubes

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Abstract

In a previous paper [1] we showed that a 3D object can be digitized without changing the topology if the object is r-regular and if the reconstruction method fulfills certain requirements. In this paper we give two important examples for such reconstruction methods. First, we introduce Majority Interpolation, an algorithm to interpolate sampling points at doubled resolution such that topological ambiguities are resolved. Second, we show how the well-known Marching Cubes algorithm has to be modified such that it is topology preserving. This is the first approach of digitizing 3D objects which guarantees topology preservation for voxel-based or polygonal surface-based reconstructions.

1 Introduction

A lot of 3D image analysis algorithms use topological information like neighborhood, connectivity, inclusion etc. Implicitly they rely on the assumption that the topological information in the digital image is the same as in the original object before digitization. But this can not be guaranteed in general. While in 2D the problem of topology preserving digitization was solved more than twenty years ago in 1982 [2, 3], a solution for the 3D case was only found recently by the authors [1]. In this paper we give two important examples of such methods. The first, Majority interpolation, is a voxel-based representation on a grid with doubled resolution. It always leads to a well-composed set in the sense of Latecki [4], which implies that a lot of problems in 3D digital geometry become relatively simple. The second method is a modification of the well-known Marching Cubes algorithm [5]. The original Marching Cubes algorithm does not always construct a topologically sound surface due to several ambiguous cases [6, 7]. We will show that most of the ambiguous cases can not occur in the digitization of an r-regular object and that the only remaining ambiguous case always occurs in an unambiguous way, which can be dealt with by a slight modification of the original Marching Cubes algorithm. Thus the generated surface is not only topologically sound, but it also has exactly the same topology as the original object before digitization.

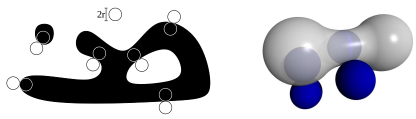

Figure 1. For each boundary point of a 2D/3D r-regular set there exists an outside and an inside osculating open disc/ball of radius r.

2 Preliminaries

All of the known topology preserving sampling theorems in 2D require the object to be r-regular [2, 3, 8]. Thus in [1] we used the 3D generalization of r-regular sets for deriving the 3D sampling theorem. A set $A \subset \mathbb{R}^3$ is called *r-regular* if, for each point $x \in \partial A$, there exist two osculating open balls of radius r to ∂A at x such that one lies entirely in A and the other lies entirely in A^c. The object is digitized using a *cubic r'-grid*, where r' is the maximal distance from a point in $\mathbb{R}^3$ to the nearest sampling point, i.e. a rotated and translated version of $\frac{2 \cdot r'}{\sqrt{3}}\mathbb{Z}^3$. The *voxels* are the disjoint cubes of sidelength $\frac{2 \cdot r'}{\sqrt{3}}$ centered in the sampling points and the *digital reconstruction* of an object is the union of voxels whose sampling points lie inside the object. One of the authors introduced the so-called *3D well-composed sets* [4], which are exactly the sets of sampling points whose digital reconstruction is a 2D manifold. Well-composed sets can be locally characterized by the fact that neither the set itself nor its complement contain one of the critical configurations (C1) and (C2) shown in Fig. 2. Now lets have a look at the 14 possible canonical configurations of eight neighboring foreground and background sampling points (see

Fig. 3), which remain after considering rotational symmetry, reflectional symmetry, and complementarity as shown by [9]. Note, that in well-composed sets only cases 1 to 7 can occur. In [8] it is shown that the digitization of an r-regular object can contain a configuration of case 8 - independently of the grid resolution (see Fig. 6(b) for an example). This means that the digital reconstruction can not be guaranteed to be well-composed – even with arbitrarily dense sampling grids. Thus, we need alternative reconstruction methods. In [1] we introduced a ball-union approach and the trilinear interpolation for that reason. But a lot of 3D image analysis algorithms need other digital representations, e.g. voxel-based or polygonal surface reconstructions. In this paper we will show that one can generate a well-composed representation by interpolating artificial sampling points at doubled resolution and that one can use a slight modification of the Marching Cubes algorithm. Due to [1] the reconstruction methods must have the following properties: (1) Any cube of configuration 1 lies completely inside or outside of the reconstructed object regarding the sampling points lying inside or outside of it. (2) Any cube/cube-pair of configuration 2 to 8 is divided by the boundary of the reconstructed object into two parts, each being homeomorphic to a ball, such that the foreground part contains exactly the foreground sampling points. Then the reconstruction method is called *topology preserving* and from [1] we know, that given an r-regular object A and a cubic r'-grid with $2r' < r$, the reconstruction result is r-homeomorphic to A.

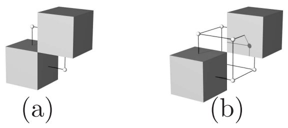

Figure 2. (a) Critical configuration (C1). (b) Critical configuration (C2). For the sake of clarity, we show only the voxels of foreground or background points.

3 Majority Interpolation

As shown in [4], a lot of difficult problems in 3D digital geometry are much easier if the images are well-composed, e.g. there exists only one type of connected component, a digital version of the Jordan-Brouwer-theorem holds and the Euler characteristic an be computed locally. There are two different approaches known to make an image well-composed: First non-deterministic changing of the voxels at positions where well-composedness is not fulfilled [10]. This approach

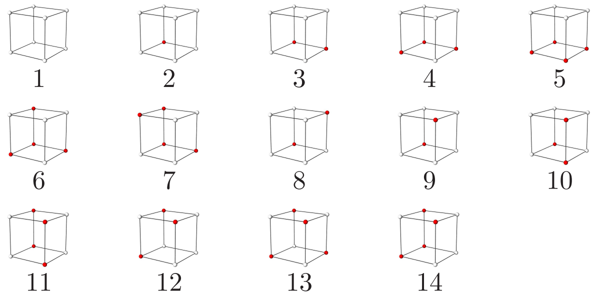

Figure 3. There are 14 distinct canonical configurations for neighboring sampling points that are either inside or outside a digitized set.

can not be used to guarantee topology preservation since it changes the image in an unpredictable way. Second, interpolating voxels on a grid with higher resolution in a well-composed way. There is a method known for the 2D case using this approach [11], which can directly be generalized to three dimensions in order to guarantee 3D well-composedness. This method uses a tripled resolution i.e. 27 times as much sampling points as in the original grid. We show that this is also possible by only doubling the resolution in any direction, i.e. using only 8 times as much voxels and give a guaranty for topology preservation for digitizations of r-regular objects.

Definition 1 *Let $A \subset \mathbb{R}^3$ be a 3D object and S a cubic sampling grid. Further let S' denote the grid of doubled resolution in any dimension containing S. A new sampling point in $S' \setminus S$ lying directly between two old ones is called a* face point*, a new sampling point lying directly between 4 face points is called* edge point*, and a new sampling point lying directly between 6 edge points is called* corner point*. Now the* majority interpolation (MI) *of A on S is the union of voxels of all sampling points $s \in S'$ fulfilling one of the following properties: (1) s is an old sampling point inside of A, or (2) s is a face point and both neighboring old sampling points are inside of A, or (3) s is an edge point and at least 4 of the 8 neighboring old sampling and face points are inside of A, or (4) s is a corner point and at least 12 of the 26 neighboring old sampling, face and edge points are inside of A.* ◇

Theorem 2 *The MI algorithm is a topology preserving reconstruction method and thus the result of the MI algorithm is r-homeomorphic to the original object if A is r-regular and the sampling grid is a cubic r'-grid with $2r' < r$.*

Proof: The well-composedness can simply be verified by checking every local configuration of 8 neighboring

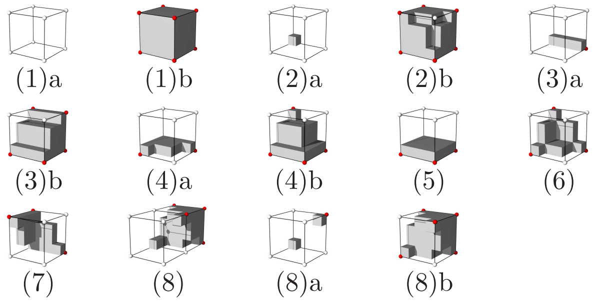

Figure 4. Cases 1 to 8 and their complements for Majority Interpolation.

old sampling points, see Fig.4. Analogously for topology preservation we have to check if in any of the 8 cases the result of the majority interpolation algorithm fulfills the requirements for a topology preserving reconstruction method. Since majority interpolation is not dual (i.e. the reconstruction of the complement of a set is different from the complement of the reconstruction of a set), we also have to check the complementary cases of the 8 configurations. Only configurations 1 to 4 differ from their complements, so here we have to consider subcases a and b. As Fig. 4 shows, the requirements are fulfilled in every of the 8+4 cases. □

By checking the rest of the 14 cases and their complements one can see that the result of the MI algorithm is always well-composed, even if the original object is not r-regular.

4 Modified Marching Cubes

One of the most common reconstruction methods is the marching cubes algorithm, introduced 1987 by Lorensen and Cline [5]. This algorithm analyses local configurations of eight neighboring sampling points in order to reconstruct a polygonal surface. Although not mentioned in the initial publication [5] the algorithm does not always produce a topologically consistent surface and might produce holes in the surface. In order to deal with these problems one has to introduce alternative configurations and decide, which of the ambiguous configurations fit together [6, 7]. Thus research has been done on how to deal with these ambiguous cases and to guarantee that the resulting surface is topologically consistent, e.g. [6, 7]. But topological consistency only means that the result is always a manifold surface – none of the proposed modifications of the marching cubes algorithm guarantees that the reconstructed surface has exactly the same topology as the original object before digitization.
The ambiguous cases of the marching cubes algorithm are exactly the cases which can not occur in a 3D well-composed image. Thus using the above presented majority interpolation algorithm to generate a well-composed image and then applying marching cubes on this new set of points would lead to a polygonal surface representation with no ambiguous cases. But this would require doubling the resolution in any dimension which leads to approximately four times as much triangular surface patches than in the original resolution. Fortunately this is not necessary: Since cases 9 to 14 can not occur in the sufficiently dense digitization of an r-regular image and since the only remaining ambiguous case 8 always occurs in a defined way, a slight modification of the original marching cubes algorithm is all we need to guarantee a reconstructed surface without any holes. Only the triangulation of the eighth case has to be changed: As already stated by Dürst [6], it is sufficient to add the two triangles making up the quadrilateral (the four intersection points along the edges of the ambiguous face, see Fig.marchingcubes(8) and old(8)). Nielson and Hamann [7] mentioned that this method may lead to edges being part of more than two triangles and thus non-manifold surfaces, but this does not happen for the only possible occurrence of case 8. Since we need only one quadrilateral in such a configuration we simply have to differentiate between the complementary parts of configuration 8 and add the quadrilateral (i.e. two triangles) only to the list of triangles of one of the two parts. In the following this slight modification of the original marching cubes algorithm will be called *modified marching cubes (MMC).*

Theorem 3 *The result of the MMC algorithm is r-homeomorphic to the surface of the original object if A is r-regular and the sampling grid is a cubic r'-grid with $2r' < r$.*

Proof: As can be seen in Fig. 5, the MMC surface divides in any of the eight cases the cube/doublecube region into two parts, one containing all foreground and one containing all background sampling points (except of the first case where there is only one such part). If one fills the foreground part, one gets a volume reconstruction method which is topology preserving. Since the MMC result is just the surface of such a reconstruction, it has to be r-homeomorphic to the surface of the original object. □

5 Conclusions

We have analysed the problems of topology preservation during digitization of r-regular objects in 3D. We showed for two reconstruction methods that they

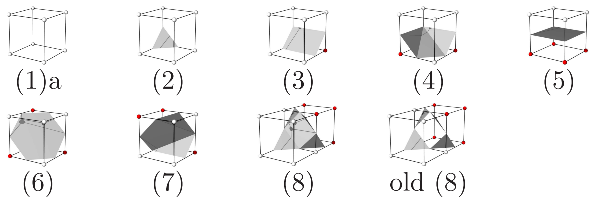

Figure 5. Cases 1 to 8 for the MMC algorithm and case 8 for the original Marching Cubes algorithm.

always lead to exactly the same topology as in the original object before digitization if this object is r-regular and the sampling grid is a cubic r'-grid with $2r' < r$. The first presented method is suitable for voxel-based approaches. Since the straightforward voxel reconstruction can not be guaranteed to be topologically correct, we introduced Majority Interpolation, a method to interpolate new voxels at doubled resolution such that the topology is always well-defined and in case of r-regular objects even identical to the original topology. Since the resulting digital object is always well-composed, several 3D digital geometry problems are much simpler. We also modified the Marching Cubes algorithm such that the generated surface has exactly the same topology as the original surface. This is the first modification of the Marching Cubes algorithm which guarantees a surface with exactly the same topology as the original object instead of only a topologically sound surface. .

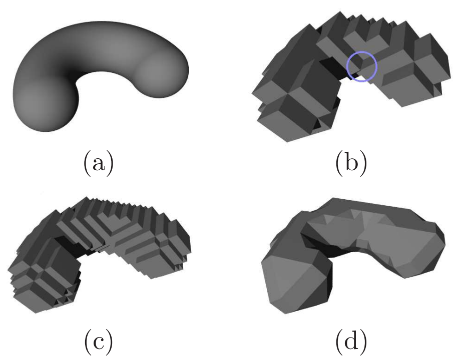

Figure 6. Digitization of an r-regular object (a) with a cubic $\frac{1}{2}r$-grid. (b) digital reconstruction (Note that the surface is not a manifold inside the circle) (c) Majority Interpolation,(d) modified Marching Cubes.

Acknowledgment

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References

[1] P. Stelldinger and L.J. Latecki. 3D Object Digitization: Topology Preserving Reconstruction. In *Submitted to the International Conference on Image Analysis, ICPR 2006.*

[2] T. Pavlidis. *Algorithms for Graphics and Image Processing.* Computer Science Press, 1982.

[3] J. Serra. *Image Analysis and Mathematical Morphology.* Academic Press, 1982.

[4] L.J. Latecki. 3D Well-Composed Pictures. *Graphical Models and Image Processing*, 59(3):164–172, 1997.

[5] W.E. Lorensen and H.E. Cline. Marching cubes: A high resolution 3D surface construction algorithm. In *SIGGRAPH '87: Proceedings of the 14th annual conference on Computer graphics and interactive techniques*, pages 163–169, New York, NY, USA, 1987. ACM Press.

[6] M.J. Dürst. Letters: Additional Reference to "Marching cubes". *SIGGRAPH Computer Graphics*, 22(2):243, 1988.

[7] G.M. Nielson and B. Hamann. The Asymptotic Decider: Resolving the Ambiguity in Marching Cubes. In *Proceedings of the 2nd IEEE Conference on Visualization (Visualization'91)*, pages 83–91, San Diego, California, USA, October 22-25 1991.

[8] P. Stelldinger and U. Köthe. Towards a General Sampling Theory for Shape Preservation. *Image and Vision Computing Journal, Special Issue on Discrete Geometry for Computer Imagery*, 23(2):237–248, 2005.

[9] A. Lopes and K. Brodlie. Improving the Robustness and Accuracy of the Marching Cubes Algorithm for Isosurfacing. *IEEE Transactions on Visualization and Computer Graphics*, 9(1):16–27, 2003.

[10] M. Siqueira, L.J. Latecki, and J. Gallier. Making 3D binary digital images well-composed. In *Proceedings of the IS&T/SPIE Conf. Vision Geometry XIII, SPIE Vol. 5675*, pages 150–163, San Jose, USA, January 2005.

[11] A. Rosenfeld, T. Kong, and A. Nakamura. Topology-preserving deformations of two-valued digital pictures. *Graphical Models and Image Processing*, 60(1):24–34, 1998.