3D Object Digitization: Topology Preserving Reconstruction

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Abstract

In this paper we derive a sampling theorem, which is the first one to guarantee topology preservation during digitization of 3D objects. This new theorem is applicable to several reconstruction methods, e.g. a union-of-balls reconstruction and the trilinear interpolation.

1 Introduction

Already two of the first books in computer vision deal with the relation between the continuous object and its digital images obtained by modeling a digitization process. Pavlidis [1] and Serra [2] proved independently in 1982 that an r-regular continuous 2D set S and the continuous analog of the digital image of S are homeomorphic, respectively have the same homotopy tree. Pavlidis used 2D square grids and Serra used 2D hexagonal sampling grids. An analog result in 3D case remained an open question for over 20 years. Only recently one of the authors proved together with Köthe that the connectivity properties are preserved when digitizing a 3D r-regular object with a sufficiently dense sampling grid [3]. But the preservation of connectivity is much weaker than of topology. In this paper we provide the solution to this problem. We use the same digitization model as Pavlidis used and we also use r-regular sets (but in $\mathbb{R}^3$) to model the continuous objects. As shown in [3] the generalization of Pavlidis' straightforward reconstruction method to 3D fails since the reconstructed surface may not be a 2D manifold. For example, Fig. 3(a) and (b) shows a continuous object and its 3D reconstruction whose surface is not a 2D manifold. However, as we will show it is possible to use other reconstruction methods that result in a 3D object with the 2D manifold surface homeomorph to the surface of the continuous 3D object.

2 Preliminaries

Definition 1 *A set $A \subset \mathbb{R}^3$ is called r-*regular *if, for each point $x \in \partial A$, there exist two osculating open balls of radius r to ∂A at x such that one lies entirely in A and the other lies entirely in A^c.* ◇

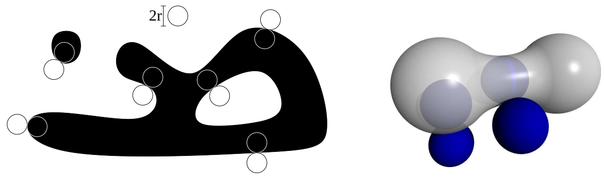

Figure 1. For each boundary point of a 2D/3D r-regular set there exists an outside and an inside osculating open disc/ball of radius r.

Note, that the boundary of a 3D r-regular set is a 2D manifold surface.

Any set S which is a translated and rotated version of the set $\frac{2 \cdot r'}{\sqrt{3}}\mathbb{Z}^3$ is called a *cubic r'-grid* and its elements are called *sampling points*. Note that the Euclidean distance $d(x,p)$ from each point $x \in \mathbb{R}^3$ to the nearest sampling point $s \in S$ is at most r'. The *voxel* $\mathcal{V}_S(s)$ of a sampling point $s \in S$ is its *Voronoi region* $\mathbb{R}^3$: $\mathcal{V}_S(s) = \{x \in \mathbb{R}^3 \mid d(x,s) \leq d(x,q),\ \forall q \in S\}$, i.e., $\mathcal{V}_S(s)$ is the set of all points of $\mathbb{R}^3$ which are at least as close to s as to any other point in S. In particular, note that $\mathcal{V}_S(s)$ is a cube whose vertices lie on a sphere of radius r' and center s.

Definition 2 *Let S be a cubic r'-grid, and let A be any subset of $\mathbb{R}^3$. The union of all voxels with sampling points lying in A is the* digital reconstruction of A with respect to S, $\hat{A} = \bigcup_{s \in (S \cap A)} \mathcal{V}_S(s)$. ◇

This method for reconstructing the object from the set of included sampling points is the 3D generalization of the 2D *Gauss digitization* (see [4]) which has been used by Gauss to compute the area of discs.

3 Digital Reconstruction of r-Regular Sets

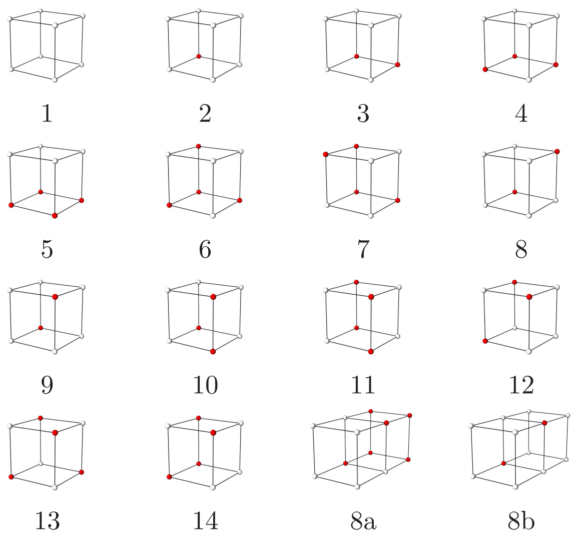

Figure 2. There are 14 different cases of canonical configurations. In dense digitizations of r-regular objects cases 9 to 14 can not occur and case 8 only occurs in complementary (8a) and not in equal pairs (8b).

Let $A \subset \mathbb{R}^3$ be an r-regular object, let S be a cubic r'-grid, and consider the digital reconstruction $\hat{A}$ of A with respect to S. Assume that no sampling point of S lies on ∂A. This assumption is not a restriction, as if some sampling point lies on ∂A, there always exists an $\varepsilon > 0$ such that the ε-opening $A \oplus \overline{\mathcal{B}}_\varepsilon$ is $(r - \varepsilon)$-regular with $r - \varepsilon > r'$, and $A \oplus \overline{\mathcal{B}}_\varepsilon$ has the same digital reconstruction as A.

Consider any cube in $\mathbb{R}^3$ whose (eight) vertices are points of S whose corresponding voxels share a common vertex. By our above assumption, each vertex of such a cube is either inside (i.e., a foreground point) or outside (i.e., a background point) A. So, there are at most 256 distinct configurations for a cube with respect to the binary "status" of its vertices. However, it has been shown ([5]) that up to rotational symmetry, reflectional symmetry, and complementarity (switching foreground and backgroud points), these 256 configurations are equivalent to the 14 canonical configurations in Fig. 2.

The problem of topology preserving digitization is that several of the 14 cases are ambiguous, which means that there are more than one possibilities to reconstruct the object locally – each with a different topology. This is not the case for sufficiently dense sampled r-regular objects, as shown by the following theorem:

Theorem 3 *Configurations 9 to 14 in Fig. 2 cannot occur in the digital reconstruction of an r-regular object with a cubic r'-grid with $2r' < r$, and case 8 always occurs in pairs, one configuration having 6 background voxels and the other having 6 foreground voxels (refer to Fig. 2(8a)).*

The proof has to be omitted in this paper since it is too long. It can be found in [6].

All the remaining cases can occur in the digitizations of r-regular objects with arbitrarily dense sampling grid, which is obvious for cases 1 to 7, which can even occur in the digitizations of planes (i.e. ∞-regular objects), and which was proven in [3] for configuration 8. This implies that the digital reconstruction of an r-regular set cannot be guaranteed to be well-composed (i.e. its surface is a manifold) just by sampling dense enough – in contrast to the 2D case. Thus one has to consider other reconstruction methods than the straightforward way of reconstructing the object by taking the union of voxels corresponding to the sampling points (i.e. digital reconstruction).

With the knowledge which configurations can not occur in the digitization of an r-regular image by using an r'-grid with $2r' < r$, we can derive a sampling theorem which can be applied to several reconstruction methods, which we call *topology preserving*.

Definition 4 *A reconstruction method is called* topology preserving *if it behaves in the following way (see Fig. 4):*

- *Any cube defined by the sampling points of configuration 1 contains no boundary part or the reconstructed object. The cube lies completely inside or outside of the reconstructed object regarding the sampling points lying inside or outside of it.*

- *Any cube defined by the sampling points of configuration 2 to 7 and any double cube defined by the sampling points of the pair of two complementary configurations of type 8 is divided by the boundary of the reconstructed object into two parts, each being homeomorphic to a ball (i.e. the part of the boundary lying inside the cube is homeomorphic to a disc), such that the part representing the foreground of the reconstruction contains all the sampling points of the configuration which are inside* *the original set and none of the sampling points which are outside the original set.*

◇

Theorem 5 *Let A be an r-regular object and S be a cubic r'-grid with $2r' < r$. Then the result of a topology preserving reconstruction method is r-homeomorphic to A.*

For proving the theorem we partition the space $\mathbb{R}^3$ into small parts – one for each configuration of eight neighboring sampling points – and show that each of these parts is r-homeomorphic to the reconstruction. The full proof is more than three pages long and can be found in [6].

Now we are able to define reconstruction methods, which guarantee to preserve the original topology of an r-regular object if one uses a cubic r'-grid with $2r' < r$.

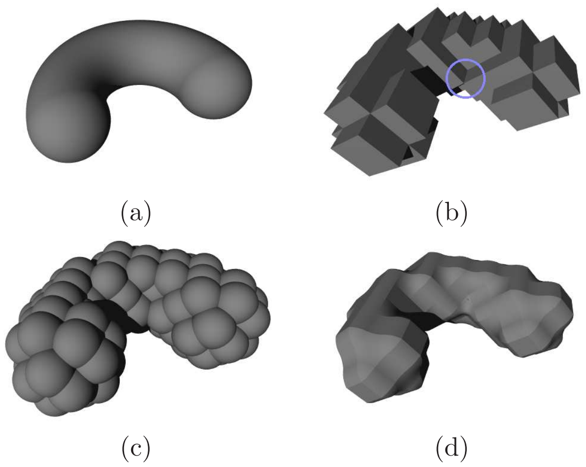

Figure 3. Digitization of an r-regular object (a) with a cubic $\frac{1}{2}r$-grid. (b) digital reconstruction (Note that the surface is not a manifold inside the circle) (c) ball union,(d) trilinear interpolation.

4 Ball Union

While the straightforward reconstruction, i.e. the union of cubical voxels does not guarantee topology preservation, we will now show that alternatively taking the union of balls with appropriate radius results in an object with correct topology. The radius of the balls has to be chosen such that the result inside any of the eight cube configurations fulfills the criterion of a topology preserving reconstruction. Thus since in case of configuration 1, when all eight sampling points are inside the sampled object, the whole cube has to be covered by the balls, their radius has to be at least r'. Otherwise the radius has to be smaller than the distance of two neighboring sampling points since a ball centered in one of the points must not cover the other. This upper bound for the radius is $\frac{2}{\sqrt{3}}r' \approx 1.155r'$. For any ball radius in between these values it can be shown that the result is topologically the same. For our illustrations we use the mean value $m = \frac{1}{2} + \frac{1}{\sqrt{3}} \approx 1.077$

Definition 6 *Let $A \subset \mathbb{R}^3$ be a binary object and S a cubic sampling grid. The* ball union (BU) *of A on S is the union of all balls $\mathcal{B}_m(s)$ with sampling points $s \in S \cap A$ and $r' < m < \frac{2}{\sqrt{3}}r'$.* ◇

Theorem 7 *The BU algorithm is a topology preserving reconstruction method and thus the result of the BU algorithm is r-homeomorphic to the original object if A is r-regular and the sampling grid is a cubic r'-grid with $2r' < r$.*

Proof: Changing m in between the given interval does not change the topology of the BU result for any of the configurations, since a topology change would require that at least two of the eight, resp. twelve sampling points have a distance d to each other with d or $2d$ being inside this interval. Thus we only have to check the eight configurations (and the complements of configurations 1 to 4 since the reconstruction is different for these subcases) for one such m. Fig. 4 shows the reconstruction for the different configurations with $m = \frac{1}{2} + \frac{1}{\sqrt{3}}$. As can be seen the requirements of a topology preserving reconstruction are fulfilled for any configuration. □

5 Trilinear Interpolation

If one wants to reconstruct a continuous object from a discrete set of sampling points, one often uses interpolation. The simplest interpolation method in 3D is the trilinear interpolation which can be seen as the combination of three linear interpolations, one for each dimension. If – as in our case – the binary information is given if a sampling point is inside or outside of the sampled object, one can assume certain grayscale values at the sampling points (i.e. 1 for the foreground and -1 for the background) and interpolate the grayscale values in between. Then Thresholding (i.e. by using the zero level set) will lead to a continuous representation of the sampled object. The interpolation result consists

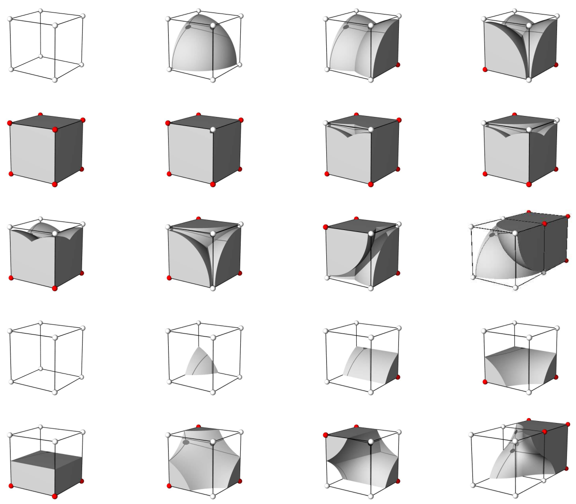

Figure 4. First and second row: cases 1 to 3 for ball union and complementary subcases; third row: cases 4 to 8 for ball union; fourth and fifth row: cases 1 to 8 for trilinear interpolation.

of smooth and nice looking patches. As we will show, the result of the trilinear interpolation of the sampled version of an r-regular object has the same topology as the original if the sampling gris is an r'-grid with $2r' < r$.

Definition 8 *Let $A \subset \mathbb{R}^3$ be a binary object and S a cubic sampling grid. Then the* trilinear interpolation (TI) *of A on S is the zero level set of a function $u : \mathbb{R}^3 \to \mathbb{R}$ with u being 1 at any sampling point inside of A, u being -1 at any sampling point outside of A and u being trilinearly interpolated between the eight sampling points of the surrounding cube C_k (see Fig. 4 for the possible cases).* ◇

Theorem 9 *The TI algorithm is a topology preserving reconstruction method and thus the result of the TI algorithm is r-homeomorphic to the original object if A is r-regular and the sampling grid is a cubic r'-grid with $2r' < r$.*

Proof: Since the trilinear interpolation inside of a cube configuration solely depends on the values at the cube corners, we only have to check the eight possible configurations. As can be seen in Fig. 4 the requirements of a topology preserving reconstruction are fulfilled for any configuration. □

6 Conclusions

We have analysed the problems of topology preservation during digitization of r-regular objects in 3D. We showed that with a sufficient sampling density several foreground-background-configurations of neighboring sampling points are not possible. We used this to derive the first sampling theorem for topology preserving digitization in 3D. Since this theorem is not restricted to a certain method for digital reconstruction, we introduced several different methods which do all fulfill the requirements of the sampling theorem. That makes our theorem directly applicable to a large variety of approaches. Since the straightforward voxel reconstruction can not be guaranteed to be topologically correct, we replaced the cubical voxels by balls of appropriate radius and proved that this is enough to guarantee topology preservation, and as an alternative we showed that the trilinear interpolation also fulfills the requirements of the sampling theorem (see Fig.3).

Acknowledgment

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