DIGITIZATIONS PRESERVING SHAPE

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Abstract. We show in this paper how it is possible to digitize a wide class of planar compacta (i.e., bounded and closed subsets of the plane) in such a way that their digitizations have the same shape, in the sense of Borsuk, as the original set. This class is formed by all those compacta having the shape of finite polyhedra. As a corollary we obtain that for a still wide subclass, the homotopy properties are also preserved under appropriate digitizations. Moreover, we show that if a set does not have the shape of a finite polyhedron, there is no possible digitization that is shape preserving.

Key Words: Shape, homotopy, approximation, polyhedra, digitizations.

1. Introduction

The Theory of Shape is a generalization of the Theory of Homotopy, introduced by K.Borsuk(1) in 1968. The goal of both theories is to analyze the global structure of spaces but, while the latter is only completely satisfactory for spaces with good local properties, the former is particularly useful when it is applied to spaces with complicated local behavior.

The basic idea behind the theory of shape for compacta is to approximate compact metric spaces with neighborhoods of them in suitable ambient spaces, like a product of n intervals or, more generally, the Hilbert cube. One of the examples in the origin of shape theory is the Warsaw circle W represented in Figure 1.

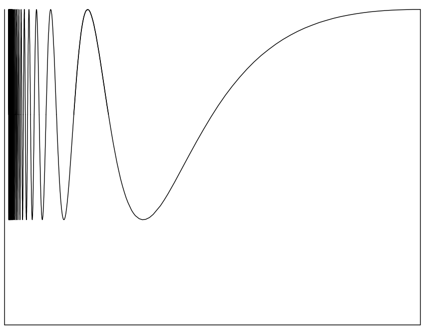

Figure 1. The Warsaw circle

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It is formed by the closure of the function $\sin(\frac{2\pi}{x})$ in $[0,4]$ and an arc closing it in a circle-like form. Although W is not a simple closed curve, it has some properties in common with a Jordan curve. In particular, it decomposes the plane in two connected components. A reason for this similarity is that any neighborhood approximating W is an annulus, which has the homotopy type of a circumference.

This process of approximation closely resembles the way in which the digitization of a set is made. In this process, the set is replaced by a covering of squares, through which the digital structure is constructed. Hence in both processes – shape through approximations, and digitizations – the local complexities are ignored. We will show that, while for "simple" sets the homotopy type is preserved under digitization, for locally complex sets this homotopy type is not invariant. If we want to know what global information is preserved in a more general setting, we must consider not the homotopy properties, but the shape properties. We will show how to apply the mathematical theory of shape to the problem of determining topological conditions under which planar sets can be digitized, and how a convenient digitization can be achieved in such a way that the shape properties are preserved under this digitization.

The idea of applying shape theory to Computer Graphics has been repeatedly proposed by M.Pavel since 1982. In particular, M.Pavel proposed using the mathematical theory of shape to model the notion of shape in Pattern Recognition[(2)].

We finally mention that J.M.R.Sanjurjo and one of the authors have recently developed in [(3)] a characterization of Borsuk's shape theory in discrete terms, opening the way to the application of combinatorial techniques in shape theory.

For information concerning shape theory, we recommend the books by K.Borsuk[(4)], J.M.Cordier and T.Porter[(5)], J.Dydak and J.Segal[(6)], and S.Mardešić and J.Segal[(7)].

For a simpler introduction directed to researchers in Computer Graphics, the reader is referred to the book by M.Pavel[(2)].

2. A Short Account of Shape Theory

We recall in this section some basic notions of the Theory of Shape as introduced by K.Borsuk[(1)]. The reader is assumed to have some familiarity with the basic notion of Homotopy Theory, as used in Digital Topology.

2.1. **Basic notions of the Theory of Retracts.** A subset Y of a space Z is said to be a retract of Z if there exists a (continuous) map $r : Z \longrightarrow Y$ (called a retraction of Z to Y) such that $r\mid_Y = \mathrm{Id}_Y$. If moreover r is homotopic in Z to Id_Z then r is said to be a deformation retract.

A metric space Y is said to be an AR-space if for every homeomorphism h mapping Y onto a closed subset $h(Y)$ of a space Z, then $h(Y)$ is a retract of Z.

A metric space Y is said to be an ANR-space if for every homeomorphism h mapping Y onto a closed subset $h(Y)$ of a space Z, there is a neighborhood U of $h(Y)$ in Z such that $h(Y)$ is a retract of U.

Some basic properties of ARs and ANRs needed throughout the paper are the following:

- Every AR is an ANR,
- The Hilbert cube, Q, and $\mathbb{R}^n$ are AR-spaces,
- every finite polyhedron is a compact ANR-space, although there exist simple polyhedra, like a hollow square (or, more generally, any closed polygonal curve), which are not ARs,
- every open subset of an ANR-space is an ANR-space,
- every ANR-space Y is an absolute neighborhood extensor, i.e. if X is a closed subset of a space X' and $f : X \longrightarrow Y$ is a continuous function, then f can be extended to a continuous function $f' : U \longrightarrow Y$ defined in a neighborhood U of X in X'.
- every ANR-space satisfies the following homotopy extension property: Let X be a closed subset of a space X' and let $Y \in$ ANR. If $H : X \times [0,1] \longrightarrow Y$ is a homotopy joining two maps $f, g : X \longrightarrow Y$ and if f has a continuous extension $f' : X' \longrightarrow Y$, then H can be extended to a homotopy $H' : X' \times [0,1] \longrightarrow Y$ joining f' with a extension $g' : X' \longrightarrow Y$ of g.

The proof of all these facts can be found in the books by K.Borsuk[1,8] and T.S.Hu[9].

2.2. **Basic concepts of Shape Theory.** In the rest of the section, compacta are assumed to lie in the Hilbert cube, Q. This is not a restriction since every compact metric space can be embedded in Q. Moreover, since we will be dealing with compacta in $\mathbb{R}^n$, we could replace Q with $\mathbb{R}^n$, and in particular with $\mathbb{R}^2$ for the case of planar compacta.

In a more general setting, the ambient space can be any absolute retract (AR) or even an absolute neighborhood retract (ANR).

Let X and Y be metric compacta. An approximative sequence from X to Y is a sequence of continuous functions $\{f_n : X \longrightarrow Q\}$ such that for every neighborhood V of Y in Q there exists $n_0 \in \mathbb{N}$ such that f_n and f_{n+1} are homotopic in V for every $n \geq n_0$.

Example 1. An approximative sequence from the circle S^1 to the Warsaw Circle W is constructed (see Figure 2) by considering a sequence of continuous functions from S^1 to $\mathbb{R}^2$ such that the images of the maps in the sequence $\{f_n\}$ are contained in progressively smaller neighborhoods of W, turning around W the same number of times (for every $n \geq n_0$, for some $n_0 \in \mathbb{N}$).

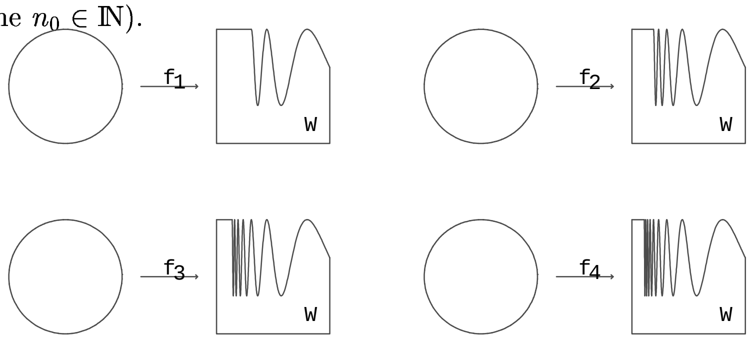

FIGURE 2. An approximative map from the circle to the Warsaw circle

Obviously, a continuous function from X to Y defines an approximative sequence from X to Y.

Two approximative sequences $\{f_n : X \longrightarrow Q\}$ and $\{g_n : X \longrightarrow Q\}$ from X to Y are homotopic if for every neighborhood V of Y in Q there exists $n_0 \in \mathbb{N}$ such that f_n and g_n are homotopic in V, for every $n \geq n_0$. We denote by $[\{f_n\}]$ the homotopy class whose elements are all the approximative sequences homotopic to $\{f_n\}$.

If we consider again the spaces in Figure 2, two approximative maps from the circle to the Warsaw circle are homotopic if they turn around the hole of W the same number of times (for every $n \geq n_0$).

Given X, Y, Z metric compacta, given $\{f_n : X \longrightarrow Q\}$ approximative sequence from X to Y and $\{g_n : Y \longrightarrow Q\}$ approximative sequence from Y to Z, the compositions $g_n f_n$ are not defined, in general. In order to define a composition $[\{g_n\}][\{f_n\}]$ we need to construct a sequence $\{g'_n : Q \longrightarrow Q\}$, where each map g'_n is an extension of the corresponding map g_n. Moreover, for the sequence $\{g'_n f_n\}$ to be an approximative map we need that $\{g'_n\}$ satisfies the following: "for every neighborhood V of Y in Q there exists a neighborhood U of X in Q and there exists $n_0 \in \mathbb{N}$ such that $g'_n|_U$ and $g'_{n+1}|_U$ are homotopic in V, for every $n \geq n_0$". This is equivalent to saying that $\{g'_n\}$ is a fundamental sequence (the Theory of Shape was, in fact, originally introduced in terms of these maps). The existence of a sequence of extensions with this property is guaranteed by the properties of the ANR-spaces stated in 2.1.

Finally, the maps in the approximative sequence $\{g'_n f_n\}$ depend on the particular extensions considered. However, the homotopy class $[\{g'_n f_n\}]$ is independent of the extensions considered. This fact allows us to define the composition $[\{g_n\}][\{f_n\}]$ as the class $[\{g'_n f_n\}]$.

2.3. **The shape of compacta.** With the tools just introduced the shape of compacta can be characterized in the following manner.

Two compacta X and Y are said to have the same shape, denoted by $Sh(X) = Sh(Y)$, if there exist an approximative sequence $\{f_n : X \longrightarrow Q\}$ from X to Y and an approximative sequence $\{g_n : Y \longrightarrow Q\}$ from Y to X such that $[\{g_n\}][\{f_n\}] = [\{i_{X,Q}\}]$ and $[\{f_n\}][\{g_n\}] = [\{i_{Y,Q}\}]$.

The relation of shape equivalence is an equivalence relation used to classify the global structure of spaces, ignoring their local properties.

Example 2. If we consider again the Warsaw circle W of Figure 1, W has the shape of a circumference.

2.4. **The relationship between the Homotopy and Shape Theories.** We mentioned earlier that a continuous function from X to Y defines an approximative sequence from X to Y. Moreover, if two continuous functions from X to Y are homotopic (using Homotopy Theory), then the induced approximative sequences from X to Y are homotopic (in Shape Theory). Thus, the following holds.

Proposition 1. *If two compacta have the same homotopy type, they have the same shape.*

The converse is true for a class of spaces with good local behavior, the class of ANR-spaces:

Proposition 2. *Let X and Y be compacta such that Y is an ANR. Then every homotopy class of approximative sequences from X to Y has a representative generated by a map $f : X \longrightarrow Y$.*

As a consequence, if X and Y are compact ANRs with the same shape, then they are homotopically equivalent.

Proof (Borsuk). Let $\{f_n : X \longrightarrow Q\}$ be an approximative sequence from X to Y. Since Y is an ANR there exist a retraction r of a closed neighborhood V of Y (in Q) to Y.

On the other hand, there exists $n_0 \in \mathbb{N}$ such that f_n and f_{n+1} are homotopic in V for every $n \geq n_0$. Consider $f : X \longrightarrow Y$ given by $f(x) = rf_{n_0}(x)$. Then the approximative map generated by f is homotopic to $\{f_n\}$. The proof of this statement (whose details can by found in the book by Borsuk[(4)]) is based on the fact (also proved in [(4)]) that the existence of the retraction r implies that for every neighborhood V' of Y in Q there exists a neighborhood V'' of Y in Q and a homotopy $H : V'' \times [0,1] \longrightarrow V'$ such that $H(y,0) = r(y)$ and $H(y,1) = y$ for every $y \in V''$.

Finally, if X and Y are compact ANRs with the same shape, then there exist approximative sequences, from X to Y and from Y to X, (generated by maps f and g, respectively) such that $\{gf\}$ is homotopic to $\{\mathrm{Id}_X\}$ and $\{fg\}$ is homotopic to $\{\mathrm{Id}_Y\}$, as approximative sequences. In the first case, we have that gf and Id_X are homotopic in any neighborhood of X. But, since X is an ANR there exist a retraction r of a closed neighborhood U of X (in Q) to X. Then, since gf and Id_X are homotopic in U we have that $rgf = gf$ and $r\,\mathrm{Id}_X = \mathrm{Id}_X$ are homotopic X. Analogously, in the second case we have that fg and Id_Y are homotopic in Y. Therefore, X and Y are homotopically equivalent. $\square$

Thus, Shape Theory is a generalization of Homotopy Theory, showing a better behavior when we want (or need) to disregard the local properties of sets. In the case of the digitization of a set, all properties under the digitization resolution are missed. This fact justifies the use of Shape Theory when studying the properties preserved under digitizations. Another proof of this applicability of Shape Theory lies in the fact that connectedness, which is one of the main invariants studied in Digital Topology, is also a shape invariant. In particular, if two compacta have the same shape, then they have the same number of connected components.

We finish this short introduction to Shape Theory citing a result which will be used later in the paper[(4)].

Theorem 1. *Two planar continua (compact connected sets) have the same shape if and only if they divide the plane in the same number of components.*

As a consequence, two finite connected polyhedra in $\mathbb{R}^2$ (being ANRs) have the same homotopy type if and only if they divide the plane in the same number of components.

3. Digitizations of Finite Polyhedra

Given a set $A \subset \mathbb{R}^2$, a r-digitization of A is a covering of A of the form

$$D_{\cap}(A) = \cup\{D_k \mid D_k \cap A \neq \emptyset\},$$

where $D = \{D_k\}$ is a grid in $\mathbb{R}^2$ of mesh r (i.e., a covering of $\mathbb{R}^2$ by closed squares of diagonal length equal to r and with the property that two different squares are else disjoint or they intersect in a proper face).

In order to see how polyhedra can be digitized in such a way that its homotopy (and hence its shape) is preserved, we will first consider the simplest case of an angle, where we call an **angle** two line segments that share a common endpoint. There exist angles L such that its digitization $D_{\cap}(L)$ has holes (see Figure 3). Thus, $D_{\cap}(L)$ is not homotopically equivalent to L.

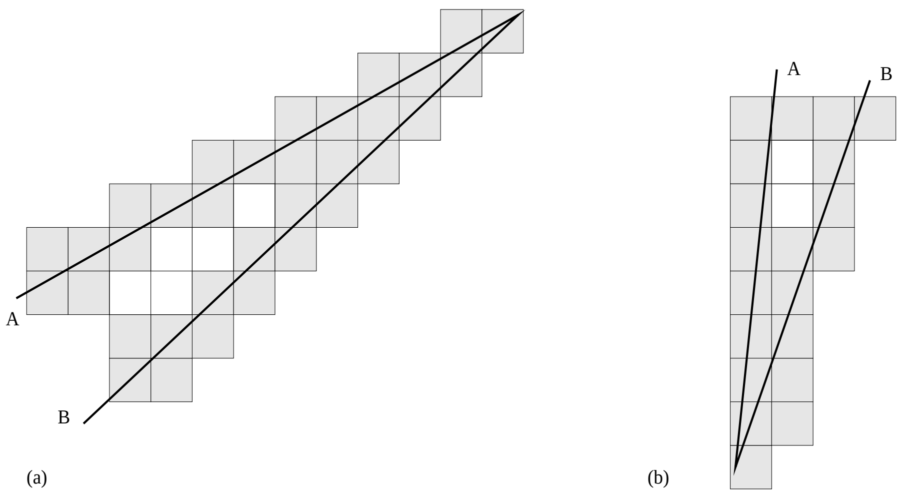

Figure 3. Digital images of angles may have holes

However, we will show for any angle L that any finite component of the background $\mathbb{Z}^2 \setminus D_{\cap}(L)$ (i.e., any white hole) cannot contain a 2×2 square composed of 4 white squares. We will call such a component of the background a **false hole**.

Proposition 3. *For every angle L, $\mathbb{Z}^2 \setminus D_{\cap}(L)$ does not contain a finite component that contains a 2×2 square.*

Proof. Let L be an angle with vertex A and line segments m and n such that $\{A\} = m \cap n$. Let r be the length of the diagonal of the squares in the square grid.

If we had a false hole, then it must be contained inside angle L, i.e., the false hole must be contained in the convex hull of L. To capture a hole in the angle, we must have two adjacent pixels (i.e., squares) p and q such that $p \cap m$ and $q \cap n$ are both nonempty. In this case we will say that the hole is captured by $p \cup q$. There are two cases: (a) p and q are 8-adjacent or (b) they are 4-adjacent.

Let $B \in p \cap m$ and $C \in q \cap n$. Every line segment parallel to BC contained inside angle L that is closer to A than BC must be shorter than BC (see Figures 4 and 5).

Case (a): Assume that a 2×2 white block WB is contained inside angle L and is captured by two black 8-adjacent squares p and q. Then there exists a translation of line segment BC that is contained in WB (see Figure 4). This translation is a line segment parallel to BC contained in angle L that is closer to A than BC, a contradiction. Thus, there is no false hole inside angle L that is captured by $p \cup q$.

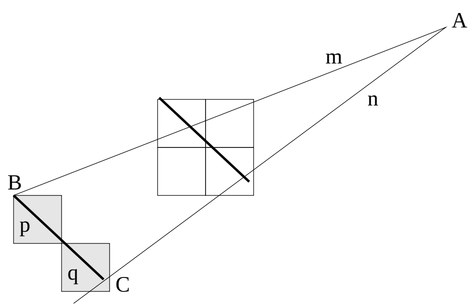

FIGURE 4. There is no false hole inside angle L that is captured by $p \cup q$

Case (b): Assume that two white 4-adjacent squares u and w are contained inside angle L and are captured by $p \cup q$, i.e., u and w are contained in a false hole captured by $p \cup q$. Then the longer sides of the rectangle $u \cup w$ cannot be parallel to the longer sides of the rectangle $p \cup q$ (see Figure 5). If this would be the case, then there exists a translation of BC that is contained in $u \cup w$. This translation is a line segment parallel to BC contained in angle L that is closer to A than BC, a contradiction.

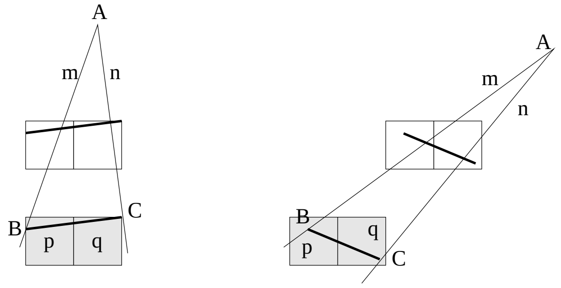

FIGURE 5. There is no false hole inside angle L that is captured by $p \cup q$

Yet, a 2×2 white block inside angle L captured by $p \cup q$ contains a rectangle $u \cup w$ with two white 4-adjacent squares u and w such that the longer sides of the rectangle $u \cup w$ are parallel to the longer sides of the rectangle $p \cup q$. Thus, there is no false hole inside angle L that is captured by $p \cup q$. In fact, this proves a slightly stronger result that there is no "L" shaped block of 3 white pixels inside angle L that is captured by two 4-adjacent black pixels. □

Example 3. To illustrate the fact that false holes can occur, consider Figure 6 showing the digitization of two "pacman" shapes.

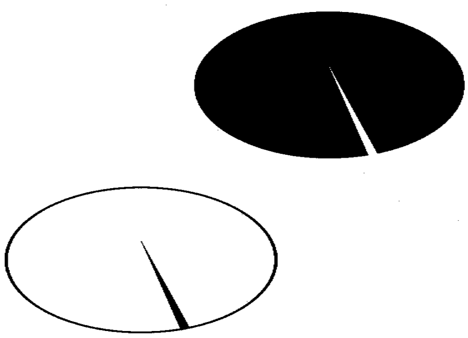

FIGURE 6. Digital images of two "pacman" shapes obtained by a scanner

This image has been generated by first producing the "pacman" shapes using the drawing program xfig. Then the print out was scanned using a HP scanner. Finally, the image obtained by the scanner was thresholded at 230, which seems to be a good approximation of intersection digitization. Figure 7 shows magnified versions of the convex part of the lower "pacman" and the concave part of the upper "pacman" in Figure 6.

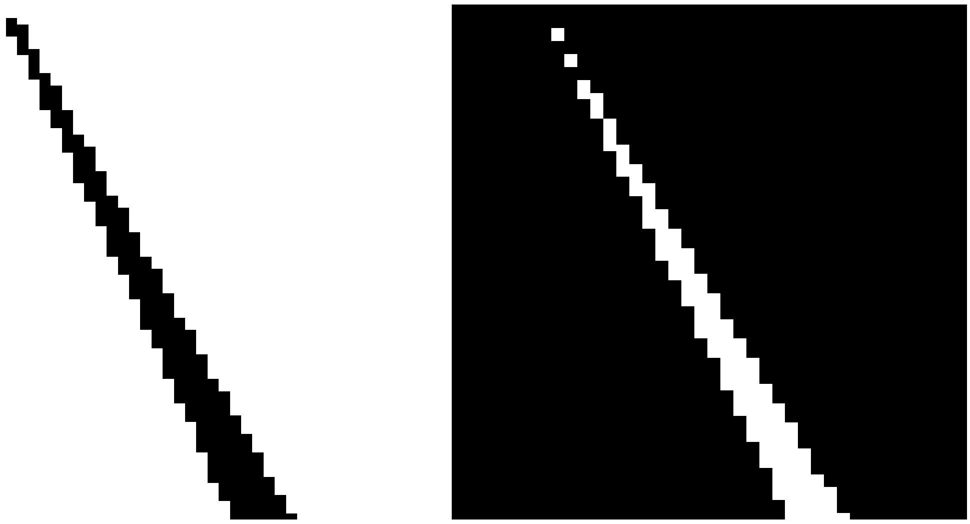

FIGURE 7. There are three false holes in digitization of the concave part

As we can see, there are three false holes in the digitization of the concave part (i.e., three white components; two are not 8-adjacent and one is not 4-adjacent to the white background), while the digitization of the convex part is clearly topology preserving.

The above proposition suggests the following post-processing operation can be used to obtain digitizations of polyhedra in such a way that their homotopy is preserved.

Definition 1. Given a finite polyhedron P in $\mathbb{R}^2$ and a digitization $D_\cap(P)$ we define $Fil(D_\cap(P))$ to be the digital set obtained after filling all holes in $D_\cap(P)$ not containing a 2×2 square.

Proposition 4. *Suppose that P is a finite polyhedron in $\mathbb{R}^2$. Then there exists $\delta > 0$ such that for any r-digitization $D_\cap(P)$ of P with $r < \delta$, then $Fil(D_\cap(P))$ and P have the same homotopy type. As a consequence, P and $Fil(D_\cap(P))$ also have the same shape.*

Proof. Suppose first that P is connected. By Theorem 1, P and its digitizations will have the same homotopy type if and only if they divide the plane in the same number of components, i.e., if and only if they have the same number of holes. Since P is contained in its digitization, we have to obtain a digitization of P in such a way that every hole of P contains one and only one hole of $Fil(D_\cap(P))$ and such that these are the only holes in $Fil(D_\cap(P))$.

To ensure that every hole of P contains a hole of $Fil(D_\cap(P))$ we consider δ_1 such that every hole in P contains a circle of diameter $3\delta_1$. Then any r-digitization $D_\cap(P)$ of P with $r < \delta_1$ satisfies the property that all holes in P contain a 2×2 square composed of 4 white squares. Therefore, these holes are preserved after filling the false holes and hence, every hole of P contains a hole of $Fil(D_\cap(P))$.

We see now how we may choose the digitization resolution in such a way that no more holes appear. We assume for a moment that P is composed of only 0 and 1-simplexes. First observe that any holes made by the digitization of two or more 1-simplexes with a common vertex are filled in $Fil(D_\cap(P))$. The appearance of any other hole is avoided taking δ_2 less than one third the minimum distance between non-adjacent 0-simplexes and between non-adjacent 0 and 1-simplexes of P, and choosing $\delta = \min\{\delta_1, \delta_2\}$. Therefore, the distance between two non-adjacent 0-simplexes or between non-adjacent 0 and 1-simplexes of P is greater than 3δ.

Assume that we have a hole H in $Fil(D_\cap(P))$ that is not contained in a hole of P. Then there must exist two 4 or 8-adjacent squares $p, q \in Fil(D_\cap(P))$ such that p and q are both 4-adjacent to H and p and q intersect two non-adjacent simplexes of P. The distance between the two simplexes intersected by p and q is $\leq 2r < 2\delta$. We obtain an inconsistency, since the two simplexes are either two non-adjacent 0-simplexes or non-adjacent 0 and 1-simplexes of P.

Thus $Fil(D_\cap(P))$ and P have the same homotopy type, for any r-digitization $D_\cap(P)$ of P with $r < \delta$. An example of such a δ is shown in Figure 8.

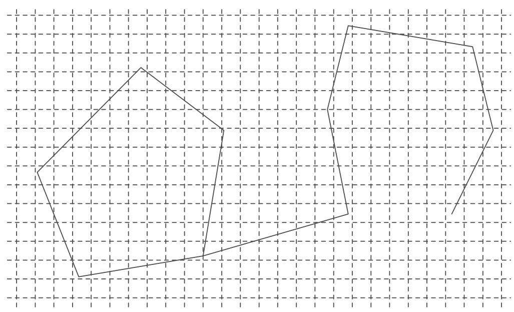

FIGURE 8. Digitization resolution of a polyhedron

If P is a connected polyhedron with simplexes of dimension 2, we take δ corresponding to its 1-skeleton following the above method. Then if we consider any r-digitization $D_\cap(P)$ of P with $r < \delta$ we have that, since any 2-simplex of P corresponds to a hole in the 1-skeleton of P, digitizing it is equivalent to filling the corresponding hole in the digitization of this 1-skeleton. Hence, taking into account the 2-simplexes of P just suppresses the same number of holes of the 1-skeleton of P and of its digitization. Therefore, P and $Fil(D_\cap(P))$ will still have the same homotopy type.

Finally, if P is not connected, we take a digitization resolution suitable for all the connected components, and lower than one third the minimum separation between components (since P is finite, there is only a finite number of such components). □

Example 4. In Proposition 4 the parameter r describes the digitization resolution that is necessary to preserve topology. We illustrate this by the two following experimental results. In Figure 9 (left), scanned at 200 dpi, the diameter of the grid squares was too large for the object (made of four letters) “s e g m”, since digital images of letters “g m” are connected. In Figure 9 (right), scanned at 400 dpi, the diameter of the grid squares is by 50% smaller, which is sufficient for topology preservation. Since both images are obtained using a real digitization device, we can also see some digitization errors.

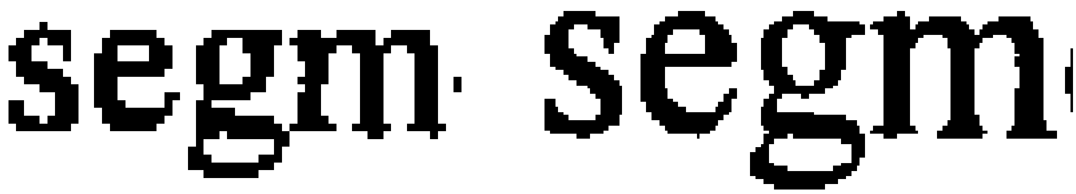

FIGURE 9. Digital images of the same object scanned at 200 dpi and 400 dpi

An analogous situation is shown in Figure 10, where the scanner resolution was 400 dpi (left) and 800 dpi (right). Assuming that the letters of a given font are modeled as polyhedra and we know the interletter spacing, Proposition 4 allows us to compute the digitization resolution necessary for topology preservation.

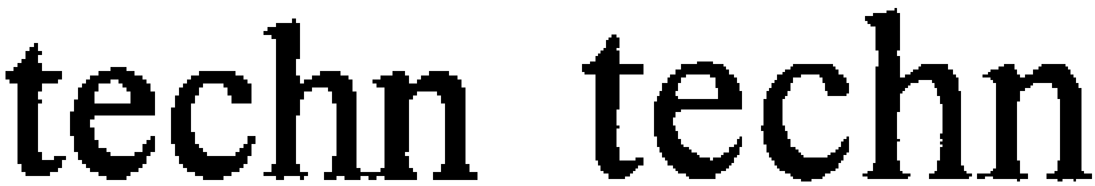

FIGURE 10. Digital images of the same object scanned at 400 dpi and 800 dpi

4. Digitizations of Simple Curves in $\mathbb{R}^2$

The above result suggests the possibility of using polyhedral approximations of compacta to digitize them. To do this, we will apply the following theorem of approximation of simple closed curves by polygonal curves. In the theorem, two sets are said to be ε-homeomorphic if there exists a homeomorphism between them such that the distance between a point and its image is always less than ε.

Theorem 2 (Bing[(11)]). *Suppose J is a simple curve in $\mathbb{R}^2$. Then for each positive number $\varepsilon > 0$ there is an ε-homeomorphism of J onto a polygon in $\mathbb{R}^2$.*

Proof. The proof of this theorem can be found in the original papers of Bing. Here, we will just sketch how to construct a polygonal approximation of J (see [(11)] for further details).

Consider a grid D of $\mathbb{R}^2$ of mesh less than $\frac{\varepsilon}{3}$ and less than $\frac{1}{3}$ the diameter of J.

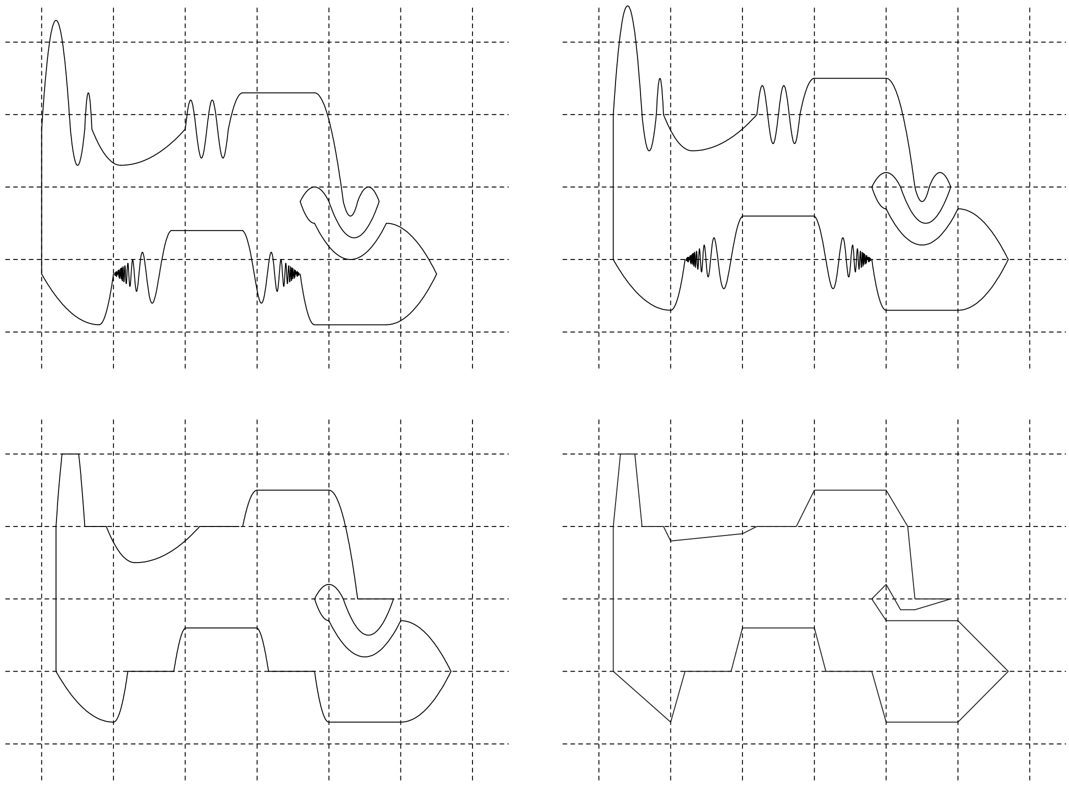

Figure 11. The three steps to find a polygonal approximation of a curve

The first step is to move J a distance lower than $\dfrac{\varepsilon}{3}$ so that it misses the 0-skeleton D_0 of the grid (intersection points of grid lines). In the second step we modify J to obtain a curve J_1 $\frac{\varepsilon}{3}$-homeomorphic to J such that $J_1 \cap D_0 = \emptyset$ and $J_1 \cap D_1$ ($D_1 =$ the 1-skeleton of the grid $=$ the union of the grid lines) has only a finite number of components. The third and last step is to replace the curve outside D_1 by polygonal curves: First, the arcs with ends in different sides of a 2-cell are replaced by straight segments. Then the arcs with ends in different sides of a 2-cell are replaced by spanning broken lines. □

Remark 1. This theorem is a simpler introduction to the theorem due to R.H.Bing[12] (see [13] for a geometric sketch of the proof), stating that any topological 2-sphere in $\mathbb{R}^3$ can be approximated by a finite polyhedron.

Consider now a curve A in $\mathbb{R}^2$. As we show in the following example, if we digitize A directly we can lose the basic structure of A, even if it is homeomorphic to a circumference.

Example 5. Consider the curve given in Figure 12.

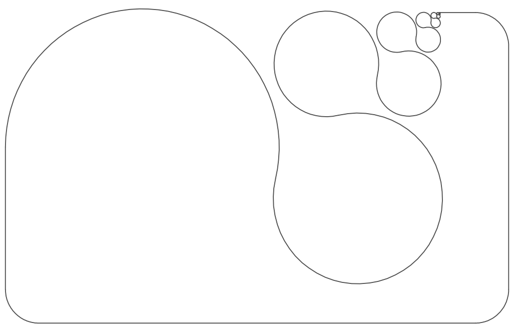

FIGURE 12. Any digitization of this simple curve has more than 1 hole

Then A is a simple curve ε-homeomorphic to a circumference, for every $\varepsilon > 0$, and hence has the shape and homotopy type of a circumference, but there exists an r such that any digitization of A with resolution smaller than r has at least two holes.

One possible approach to the problem suggested by this example is that proposed by two of the authors[10] using their notion of par(r)-regular sets (the curve in figure 12 is not par(r)-regular). We propose here a different approach which will allows us, first to digitize a polygonal approximation of any simple curve (see Theorem 3), and then as many planar sets as possible (see Theorem 4 and Remark 3), always preserving the global structure.

Theorem 3. *Let A be a simple curve in $\mathbb{R}^2$. Then for every $\varepsilon > 0$ there exists a polygonal curve $C \subset \mathbb{R}^2$ ε-homeomorphic to A and there exists a $\delta > 0$ such that if $D_\cap(C)$ is a r-digitization of C with $r < \delta$, then A, C and $Fil(D_\cap(C))$ have the same homotopy type.*

As a consequence, A, C and $Fil(D_\cap(C))$ also have the same shape.

Proof. Suppose that A is a simple closed curve in $\mathbb{R}^2$. Consider $\varepsilon > 0$. Then, by Theorem 2, there exists a polygon C ε-homeomorphic to A (hence A and C have the same homotopy type). On the other hand, by Proposition 4, there exists $\delta > 0$ such that if $D_\cap(C)$ is a r-digitization of C with $r < \delta$, then C and $Fil(D_\cap(C))$ have the same homotopy type. □

Example 6. Consider the simple curve given in Figure 12. As we mentioned before, any digitization of it at a sufficient resolution has at least one hole. In Figure 13 we show the polygonal approximation and digitization resolution (focused near the locally complicated part of the set) obtained following our method.

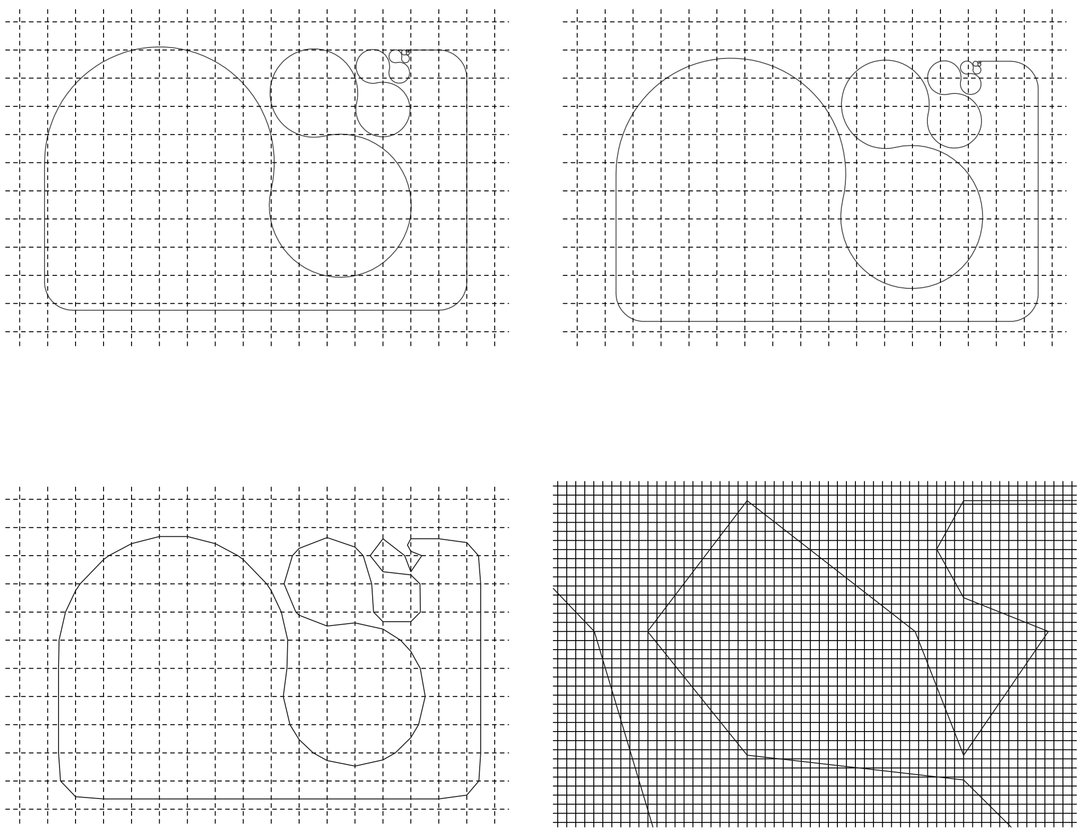

FIGURE 13. The polygonal approximation and digitization resolution of a locally complicated simple curve

In Figure 14 we compare the results of digitizing this set directly, and following our method (observe the holes in the direct digitization). To increase the resolution, we have just displayed the digitizations of the most complicated part of the set.

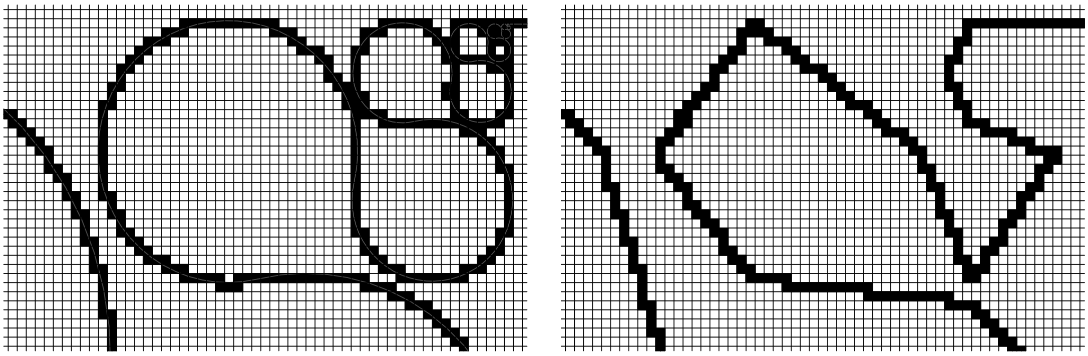

FIGURE 14. Two digitizations of a locally complicated simple curve

Remark 2. Observe that if we take r small enough, it may happen that the curve A is not covered by the digitization of its polygonal approximation C (see Figure 14). The technique developed in the next section will allow us to digitize simple curves via polygonal approximations, fulfilling also this covering condition.

5. Digitizations of Planar Compacta

In the next theorem, which is the main result of the paper, we prove that the class of compacta that can be appropriately digitized is formed precisely by those compacta having the shape (in the sense of Borsuk) of finite polyhedra. For a detailed account on polyhedra and in particular on its homotopy and shape properties see Appendix 1 in the book by Mardešić and Segal[7].

Theorem 4. *Let A be a compact subset of $\mathbb{R}^2$. The following are equivalent:*

i) A has the shape of a finite polyhedron.

ii) A has a finite number of components and each component decomposes the plane into a finite number of regions.

iii) For every $\varepsilon > 0$ (that can be seen as the resolution desired) there exists a finite polyhedron $C \subset \mathbb{R}^2$ such that $A \subset C \subset B_\varepsilon(A)$ and there exists a $\delta > 0$ (the digitization resolution required) such that if $D_\cap(C)$ is a r-digitization of C with $r < \delta$, then $Fil(D_\cap(C))$, C and A have the same shape.

Moreover, since $A \subset C \subset B_\varepsilon(A)$, then $A \subset D_\cap(A) \subset D_\cap(C) \subset Fil(D_\cap(C)) \subset B_{\varepsilon+2r}(A)$.

Proof. The equivalence of the first two statements is well-known in Shape Theory (see, for instance [7]). To prove that ii) implies iii), we may suppose that A is connected and decomposes $\mathbb{R}^2$ into $n+1$ open connected regions $G_0, G_1, \ldots, G_n$. Given $\varepsilon > 0$ there exists a continuum B with $A \subset B \subset B_{\frac{\varepsilon}{2}}(A)$ such that its boundary is the disjoint union of $n+1$ simple closed curves $B_0, B_1, \ldots, B_n$ with $B_i \subset G_i$ for every i (see (9.2) and (9.3) in [(1)]).

Since $B_i \subset G_i$ then $d(B_i, \mathbb{R}^2 \setminus G_i) > 0$. Take $0 < \varepsilon' < \min\{\varepsilon, d(B_i, \mathbb{R}^2 \setminus G_i)\}$. Then applying the method of the previous section to $B_0, B_1, \ldots, B_n$, we obtain polygonal curves $P_0, P_1, \ldots, P_n$ $\frac{\varepsilon'}{2}$-homeomorphic to $B_0, B_1, \ldots, B_n$ respectively. Moreover, since $\varepsilon' < d(B_i, \mathbb{R}^2 \setminus G_i)$, we have that $P_i \subset G_i$ for every i.

Since $P_i \subset G_i$ for every i, A is contained in a bounded connected component C of $\mathbb{R}^2 \setminus \cup_{i=0}^n P_i$, whose boundary is made of the $n+1$ closed polygonal curves $P_0, P_1, \ldots, P_n$. Hence C decomposes $\mathbb{R}^2$ in $n+1$ regions. Therefore A and C have the same shape and, since C is a finite polyhedron, then there exists a $\delta > 0$ such that $Sh(C) = Sh(Fil(D_\cap(C))$ for any r-digitization of C with $r < \delta$. Finally, since ∂C is contained in $B_\varepsilon(A)$, so is C.

Finally, the fact that $A \subset C$, easily implies that $A \subset D_\cap(A) \subset D_\cap(C) \subset Fil(D_\cap(C))$. On the other hand, $D_\cap(C) \subset \bar{B}_r(C)$ (where $\bar{B}_r(C)$ stands for the closed ball of radius r) and this implies, since any point in a false hole is at a distance lower or equal than r from $D_\cap(C)$, that

$$Fil(D_\cap(C)) \subset \bar{B}_r(D_\cap(C)) \subset \bar{B}_{2r}(C) \subset \bar{B}_{2r}(B_\varepsilon(A)) \subset B_{\varepsilon+2r}(A).$$

Finally, iii) implies i) in an obvious way. □

Remark 3. In the above theorem we have shown how to digitize compacta with the shape of polyhedra in such a way that their shapes are preserved. On the other hand, since any digitization of a set – made through any possible method – is made from the faces of a grid, it must have the shape of a finite polyhedron. Therefore, if a set does not have the shape of a finite polyhedron, it is not possible to digitize it by any method, in such a way that its shape is preserved.

Corollary 1. *Let A be a compact ANR contained in $\mathbb{R}^2$. Then:*

i) A has the homotopy type of a finite polyhedron.

ii) For every $\varepsilon > 0$ there exists a finite polyhedron $C \subset \mathbb{R}^2$ such that $A \subset C \subset B_\varepsilon(A)$ and there exists a $\delta > 0$ such that if $D_\cap(C)$ is a r-digitization of C with $r < \delta$, then $Fil(D_\cap(C))$, C and A are homotopically equivalent. Moreover, $A \subset D_\cap(A) \subset D_\cap(C) \subset Fil(D_\cap(C)) \subset B_{\varepsilon+2r}(A)$.

Proof. i) is a well-known result in Shape Theory (see, for example, Theorem 5 in page 317 of [7]). ii) is an immediate consequence of Theorem 4, Proposition 2 and the fact that a digitization of a space is a compact ANR-space. □

Example 7. If we consider the Warsaw circle W represented in Figure 1, then W has the shape (but not the homotopic type) of a circumference. Therefore, the digitization developed in this paper will yield a set with the shape of a circumference. Moreover, since this digitization is a compact ANR-space, then it will also have the homotopy type of a circumference. Hence for W, the homotopy type is not preserved under our digitization (the same happens for a direct digitization). The reason for this is that W is not an ANR.

6. Conclusions. The Higher Dimensional Case

We have presented in this paper a first connection between two (until now) unrelated areas: the mathematical Theory of Shape and Digital Topology, which deals with topological properties of discrete objects in Computer Graphics and Computer Vision. Summarizing, we have shown that, for a planar set A,

i) if A is a finite polyhedron then A can be digitized and its homotopy type is preserved,
ii) if A is a "simple enough" set (compact ANR-set), then A can be digitized using a finite polyhedral approximation containing A, and its homotopy type is preserved,
iii) if A is any compactum with the shape of a finite polyhedron then A can be digitized using a finite polyhedral approximation containing A, and its shape is preserved,
iv) if A does not have the shape of a finite polyhedron, then it is not possible to digitize A by any method, in such a way that its shape preserved.

The use of Shape Theory when studying properties preserved under digitization, is justified by the following analogy:

- When the shape of a set is computed, all local properties are ignored.
- When a set is digitized, all properties under the digitization resolution are missed.

We propose finally the following extensions of the results of this paper to higher dimensions. The operator Fil^* would be a higher dimensional analogous of the filling operator introduced in this paper for the planar case. Note that the implication ii) $\Rightarrow$ i) in Conjecture 1 is obviously true. For Conjecture 2: i) is a well-known result in Shape Theory (see, for example, Theorem 5 in page 317 of [7]) while ii) would be an immediate consequence of Conjecture 1.

Conjecture 1. *Let A be a compact subset of $\mathbb{R}^n$. Then following are equivalent:*

i) *A has the shape of a finite polyhedron.*

ii) *For every $\varepsilon > 0$ there exists a finite polyhedron $C \subset \mathbb{R}^n$ such that $A \subset C \subset B_\varepsilon(A)$ and there exists a $\delta > 0$ such that if $D_\cap(C)$ is a r-digitization of C with $r < \delta$, then $Sh(Fil^*(D_\cap(C))) = Sh(C) = Sh(A)$. Moreover, $A \subset D_\cap(A) \subset D_\cap(C) \subset Fil^*(D_\cap(C)) \subset B_{\varepsilon+k_n r}(A)$, where k_n depends on the upper bound for the diameter of the false holes in $\mathbb{R}^n$.*

Conjecture 2. *Let A be a compact ANR contained in $\mathbb{R}^n$. Then:*

i) *A has the homotopy type of a finite polyhedron.*

ii) *For every $\varepsilon > 0$ there exists a finite polyhedron $C \subset \mathbb{R}^n$ such that $A \subset C \subset B_\varepsilon(A)$ and there exists a $\delta > 0$ such that if $D_\cap(C)$ is a r-digitization of C with $r < \delta$, then $Fil^*(D_\cap(C))$, C and A are homotopically equivalent. Moreover, $A \subset D_\cap(A) \subset D_\cap(C) \subset Fil^*(D_\cap(C)) \subset B_{\varepsilon+k_n r}(A)$, where k_n depends on the upper bound for the diameter of the false holes in $\mathbb{R}^n$.*

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