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Digitizations Preserving Topological and Differential Geometric Properties

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In this paper, we present conditions which guarantee that every digitization process preserves important topological and differential geometric properties. These conditions also allow us to determine the correct digitization resolution for a given class of real objects. Knowing that these properties are invariant under digitization, we can then use them in feature-based recognition. Moreover, these conditions imply that only a few digital patterns can occur as neighborhoods of boundary points in the digitization. This is very useful for noise detection, since if the neighborhood of a boundary point does not match one of these patterns, it must be due to noise. Our definition of a digitization approximates many real digitization processes. The digitization process is modeled as a mapping from continuous sets representing real objects to discrete sets represented as digital images. We show that an object *A* and the digitization of *A* are homotopy equivalent. This, for example, implies that the digitization of *A* preserves connectivity of the object and its complement. Moreover, we show that the digitization of *A* will not change the qualitative differential geometric properties of the boundary of *A*; i.e., a boundary point which is locally convex cannot be digitized to a locally concave pixel and a boundary point which is locally concave cannot be digitized to a locally convex pixel. © 1995 Academic Press, Inc.

1. INTRODUCTION

Serra [11] considered different kinds of digitizations. He showed that, for a certain class of planar sets, their digitizations preserve homotopy, which implies that there is a complete correspondence between connected components of the planar set and its digitization and their complements. However, he proved this only for digitizations in hexagonal grids, where a digitization of a set in R^2 is the set of points in Z^2 which are contained in the set. To show nontrivial problems connected with digitizations, Serra gave the following title to one of the sections: "To digitize is not as easy as it looks" [11, p. 211].

We consider digitizations which are more relevant to practical applications. Our digitization, consistent with real sensor output, is defined with respect to a cover of a 2D (or 3D) set of squares (or cubes) with diameter r. A square (or cube) is a black pixel (or voxel) iff the ratio of the area (volume) of the object to the area (volume) of the field "seen" by the corresponding sensor is greater than some constant threshold value.

Assume that a real object can be represented as a closed subset A of the plane or 3D space such that its boundary bdA is compact and the second derivative exists for every point $a \in bdA$. Then the main results in this paper are as follows: (i) any real object has a digitization resolution r that preserves the topology of the object and its complement; (ii) a digitization preserving topology does not change the qualitative differential geometric properties on the boundary of a real object; i.e., a boundary point which is locally convex cannot be digitized to a locally concave pixel and a boundary point which is locally concave cannot be digitized to a locally convex pixel.

Pavlidis [8] tried to generalize to two-dimensions Shannon's Sampling Theorem, which is well known in one-dimensional signal processing. First we quote his definition of compatibility:

> "A binary image and a square sampling grid whose (square) cells have diameter h are *compatible* if: (a) There exists a number $d > h$ such that for each boundary point of each region R of a given color, there is a circle C with diameter d that is tangent to the boundary and lies entirely within R. (b) The same is also true for the complement of R" [8, Definition 7.4].

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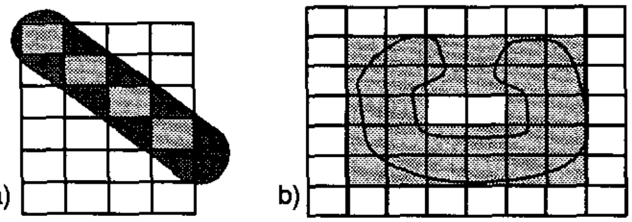

FIG. 1. The object and its digitization are not homeomorphic in (a) and not homotopy equivalent in (b).

Using this definition, Pavlidis stated the following theorem:

> "For a 2D set A, the condition of compatibility implies that A and its digitization are topologically homeomorphic" [8, Theorem 7.1].

However, this theorem is not true, as the following examples show. Pavlidis obtains a binary image by applying some threshold value to a gray-level image that is the output of some digitization process. This can be modeled by coloring an image point black if the ratio of the area of the object in a square representing the point to the area of the entire square is greater than some threshold value.

Let A be a strip of width d, where $2h > d > h$, forming a 45° angle with the square grid as illustrated in Fig. 1a. A square p is black iff $area(p \cap A)/area(p) = 1$ and white otherwise. Then the digitization of strip A represented by the gray squares is a digital 8-line, which is not homeomorphic to strip A. Note, however, that A and its digitization are homotopy equivalent. This is not the case for our second example illustrated in Fig. 1b, where a square p is black iff $area(p \cap A)/area(p) > 0$ and white otherwise. Here set A is not even homotopy equivalent to its digitization represented by gray squares, since A is simple connected, but its digitization is not simple connected (there is a white "hole" in it).

From Pavlidis' point of view, one of the main results in this paper is that we give a solution to the two- and three-dimensional "sampling problem."

2. GEOMETRIC DIFFERENTIAL GEOMETRY

In this section, we define a class of subsets of the plane or 3D space representing "real objects," which we will call par(r)-regular sets. We assume that A is a closed subset of the plane or space such that its boundary bdA is compact and, for every point $a \in bdA$, the tangent line (or plane) at a is well defined. We denote the tangent line (or plane) at a by $t(a)$ and the normal line at a by $nl(a)$. Note that A, as well as bdA, does not have to be connected. All the results in this paper, unless explicitly stated otherwise, hold in 2D as well as in 3D.

DEFINITION. For every $a \in bdA$, let $n(a, r)$ denote the *normal vector* at a of length r pointing toward the outside of A. We also view $n(a, r)$ as the set of points located on it. We denote the normal vector at a of length r pointing toward the inside of A by $-n(a, r)$.

DEFINITION. A set A will be called *par(r, +)-regular* if, for every two distinct points $x, y \in bdA$, $n(x, r)$ and $n(y, r)$ do not intersect. For example, in Fig. 2, set X is not par(r, +)-regular while set Y is par(r, +)-regular, where r is the length of the depicted vectors. A set A will be called *par(r, −)-regular* if, for every two distinct points $x, y \in bdA$, $-n(x, r)$ and $-n(y, r)$ do not intersect. A set A will be called *par(r)-regular* if it is par(r, +)-regular and par(r, −)-regular. A set A will be called *parallel regular* if there exists a constant r such that A is par(r)-regular.

We will sometimes call parallel regular sets *objects*. Assuming an object is parallel regular, we now want to define the notion of a parallel set by adding the normal vectors of fixed length to the original set.

DEFINITION. The *parallel set* of A with distance r is given by $Par(A, r) = A \cup \bigcup\{n(a, r) : a \in \mathrm{bdA}\}$. For $-r$, we have $Par(A, -r) = cl(A - \bigcup\{-n(a, r) : a \in bdA\})$, where cl denotes the usual set closure in R^2 and R^3.

DEFINITION. $B(x, r)$ denotes the closed ball of radius r centered at a point x. $Dil(A, r)$ denotes the dilation of A with balls of radius r; i.e., $Dil(A, r) = \bigcup\{B(x, r) : x \in A\}$.

In the rest of this section, we establish some basic properties of parallel regular sets.

LEMMA 1. *If A is par(r, +)-regular, then $n(a, r) \cap A = \{a\}$ for all $a \in bdA$. If A is par(r, −)-regular, then $-n(r, d) \subseteq A$ for all $a \in bdA$.*

Proof. We show that $n(a, r) \cap A = \{a\}$ for all $a \in bdA$; the proof for the second part is similar. Since vector $n(a, r)$ points toward the outside of A, it cannot be completely contained in A. Therefore, if $(n(a, r) - \{a\}) \cap A \neq \{a\}$, then there exists $x \in bdA \cap (n(a, r) - \{a\})$. However, since $x \in n(x, r)$, $n(a, r) \cap n(x, r) \neq \emptyset$. ■

LEMMA 2. *Let $x \notin A$ and let $s \in bdA \cap B(x, r)$ be a point with the shortest distance to x from all points in $bdA \cap$*

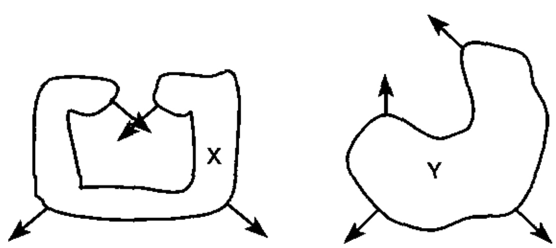

FIG. 2. X is not par(r, +)-regular, but Y is par(r, +)-regular.

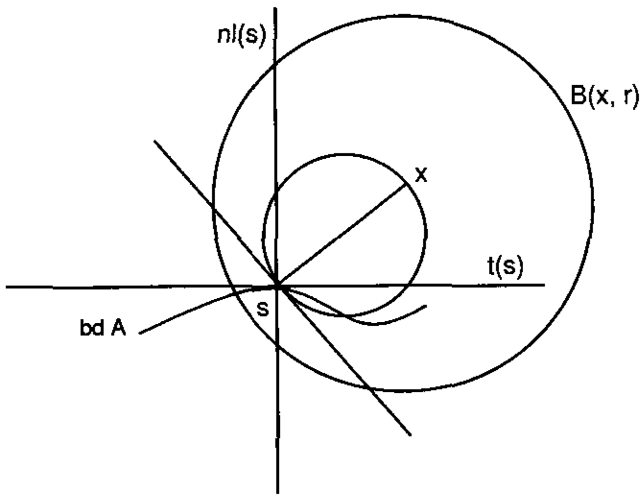

FIG. 3. x does not lie on $nl(s)$.

$B(x, r)$. *Then* $x \in n(s, r)$. *Let* $x \in A - bdA$ *and let* $s \in bdA \cap B(x, r)$ *be a point with the shortest distance to* x *from all points in* $bdA \cap B(x, r)$. *Then* $x \in -n(s, r)$.

Proof. We prove only the first part of the theorem. The proof of the second part is analogous. Let $x \notin A$ and let d be the distance from x to s. It is clear that $d \leq r$. We will show that x lies on the normal line $nl(s)$. This implies that $x \in n(s, r)$ or $x \in -n(s, r)$, since the distance between s and x is smaller than or equal to r. However, $x \notin -n(s, r)$, because there would then exist some $z \in -n(s, r) \cap bdA$ which lies strictly between s and x, since $x \notin A$. This would violate the fact that $s \in bdA \cap B(x, r)$ is the closest point to x. Thus, we obtain that $x \in n(s, r)$.

It remains to show that x lies on normal line $nl(s)$. Let us assume the contrary that x does not lie on $nl(s)$. The above considerations hold in the 2D case, as well as in the 3D case. The remaining part of the proof is based on 2D arguments. Therefore, if A is a 3D set, we consider the cut of A with the plane containing line segment xs and $nl(s)$.

Let C be the circle going through x and s with diameter being line segment xs (see Fig. 3).

Observe that the line segment xs and the tangent line $t(s)$ are not perpendicular, since $x \notin nl(s)$. Let $L(s, z)$ denote a straight line passing through points s and z. By the definition of a tangent line, the angle between $t(s)$ and $L(s, z)$ goes to zero as $z \in bdA$ goes to s. Therefore, there exists a point $y \in bdA \cap B(x, r)$ distinct from s such that y lies inside circle C. Since y lies inside C, $|xy| < |xs| = d$. This contradicts the fact that $s \in bdA \cap B(x, r)$ is a point with the shortest distance to x. Therefore, x must lie on $nl(s)$. ■

LEMMA 3. *Let* $x \notin A$. *If* $bdA \cap B(x, r) \neq \emptyset$, *then there exists* $s \in bdA \cap B(x, r)$ *such that* $x \in n(s, r)$ *and* s *is a point with the shortest distance to* x *from all points in* $bdA \cap B(x, r)$. *Let* $x \in A - bdA$. *If* $bdA \cap B(x, r) \neq \emptyset$, *then there exists* $s \in bdA \cap B(x, r)$ *such that* $x \in -n(s, r)$ *and* s *is a point with the shortest distance to* x *from all points in* $bdA \cap B(x, r)$.

Proof. We prove only the first part of the theorem. The proof of the second part is analogous. Since $bdA \cap B(x, r)$ is compact, there exists a point $s \in bdA \cap B(x, r)$ having the shortest distance to x. By Lemma 2, we obtain that $x \in n(s, r)$. ■

THEOREM 1. *The parallel set of* A *with distance* r *is equal to the dilation of* A *with radius* r, *i.e.,* $Par(A, r) = Dil(A, r)$ *(see Fig. 4).*

Proof. It is easy to note that $Par(A, r) \subseteq Dil(A, r)$; simply observe that $n(s, r)$ is contained in the dilation ball $B(s, r)$ for every $s \in bdA$.

It remains to show that $Dil(A, r) \subseteq Par(A, r)$. It is clear that $A \subseteq Par(A, r)$. So, let $a \in Dil(A, r)$ and $a \notin A$. Let $s \in bdA$ be a point having the shortest distance d from a to bdA. Such a point exists, since bdA is compact. Of course, $d \leq r$. By Lemma 3, $a \in n(s, d)$. Since $d \leq r$, we obtain $a \in n(s, r)$. Thus, $a \in Par(A, r)$. ■

The following definition generalizes the concept of the osculating circle.

DEFINITION. Let $a \in bdA$ and let $n(a, e)$ be the normal vector at a of length e pointing toward the outside of A. For every $a \in bdA$, we define an *inside osculating ball* $iob(a, e)$ at a with radius e to be the open ball centered at point $a - n(a, e)$ with radius e. For every $a \in bdA$, we define an *outside osculating ball* $oob(a, e)$ at a with radius e to be the open ball centered at point $a + n(a, e)$ with radius e.

Note that the following holds for every $z \in bdA$:

$$iob(z, e) \cap oob(z, e) = \emptyset,$$
$$t(z) \cap iob(z, e) = \emptyset, \text{ and } t(z) \cap oob(z, e) = \emptyset.$$

The following theorems justify the definition of osculating balls for parallel regular sets.

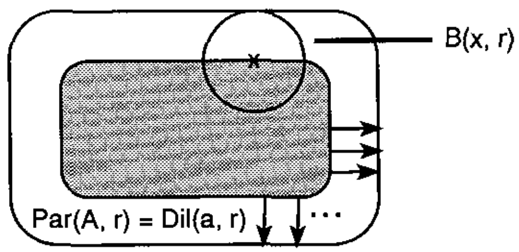

FIG. 4. The parallel set of A with distance r is equal to the dilation of A with radius r.

THEOREM 2. *Let A be par(r, +)-regular. Then $oob(x, r) \subseteq A^c$ for every $x \in bdA$, where A^c denotes the complement of A.*

Proof. By Lemma 1, if $a \in bdA$, then point $a + n(a, r) \notin A$. We show that $oob(x, r) \cap bdA = \emptyset$. Assume that this is not the case, i.e., $int\ B(a + n(a, r), r) \cap bdA \neq \emptyset$. Therefore, there exists $r' < r$ such that $B(a + n(a, r), r') \cap bdA \neq \emptyset$. By Lemma 3, there exists $s \in bdA \cap B(a + n(a, r), r')$ such that $a + n(a, r) \in n(s, r')$. But this means that $n(a, r) \cap n(s, r) = \emptyset$, which contradicts the fact that A is par(r, +)-regular. Thus, $oob(x, r) \cap bdA = \emptyset$. Since $n(a, r)$ points toward the outside of A, we obtain that $oob(x, r) \subseteq A^c$. ∎

THEOREM 3. *Let A be par(r, −)-regular. Then $iob(x, r) \subseteq A$ for every $x \in bdA$.*

Proof. If A is par(r, −)-regular, then clA^c is par(r, +)-regular, where clA^c denotes the closure of the complement of A. By Theorem 2, it follows that, for every $x \in bdA$, $oob(x, r) \subseteq cl(clA^c)^c = A$. Since $oob(x, r)$ with respect to A^c is just $iob(x, r)$ with respect to A, we obtain that $iob(x, r) \subseteq A$. ∎

THEOREM 4. *Let A be par(r)-regular. Then $iob(x, r) \subseteq A$ and $oob(x, r) \subseteq A^c$ for every $x \in bdA$.*

Proof. It follows from Theorems 2 and 3. ∎

THEOREM 5. *Let A be a closed subset of the plane or space with a compact boundary such that the second derivative exists at every point $x \in bdA$ and is continuous. Then there always exists $r > 0$ such that A is par(r, +)-regular.*

Proof. The proof is constructive: we calculate a constant $r > 0$ such that A is par(r, +)-regular; i.e., for every $x, y \in bdA$, $n(x, r)$ and $n(y, r)$ do not intersect.

Step 1. Let $k_{\max}$ be the maximum of the absolute values of the principal curvatures at every point on bdA (the existence follows from compactness of bdA).

Step 2. By elementary arguments from differential geometry (see [1, 12]), it follows that, for every $t > 1/k_{\max}$ and every $x \in bdA$, there exists $e(x) > 0$ such that, for all $y \in bdA$, $d_{bdA}(x, y) < e(x)$ implies that $n(x, t)$ and $n(y, t)$ do not intersect, where d_{bdA} is the intrinsic distance on bdA.

Step 3. Compactness of $bd\,A$ implies that there exists $\varepsilon = \min\{e(x): x \in bdA\}$ such that $\varepsilon > 0$ and

$$(\forall t < 1/k_{\max}\ \forall x \in bdA\ \forall y \in bdA)(d_{bdA}(x, y) < e(x) \text{ implies that } n(x, t) \cap n(y, t) = \emptyset).$$

Step 4. For every $x \in bdA$, let $d(x) = \min\{d(x, y): y \in bdA \text{ and } d_{bdA}(x, y) \geq \varepsilon\}$, where d is the Euclidean distance in R^2 (or R^3). Since the tangent at x is well defined, $d(x) > 0$. Let $d_{\min} = \min\{d(x): x \in bdA\}$. By compactness of bdA, $d_{\min}$ exists and $d_{\min} > 0$.

Step 5. Let $r < \min\{d_{\min}/2, 1/k_{\max}\}$. For every $x, y \in bdA$, $n(x, r)$ and $n(y, r)$ do not intersect: Let $x \in bdA$. From Step 3, it follows that, for every $y \in bdA$ such that $d_{bdA}(y, x) < \varepsilon$, $n(x, r)$ and $n(y, r)$ do not intersect. From Step 4, it follows that, for every $y \in bdA$ such that $d_{bdA}(y, x) \geq \varepsilon$, $n(x, r)$ and $n(y, r)$ do not intersect, since $d(x, y) \geq d_{\min} > 2r$. Thus, A is par(r, +)-regular. ∎

THEOREM 6. *Let A be a closed subset of the plane or space with a compact boundary such that the second derivative exists at every point $x \in bdA$. Then A is parallel regular.*

Proof. By Theorem 5, there exists a constant r_1 such that A is par(r_1, +)-regular. Applying this theorem to clA^c, we obtain that there exists a constant r_2 such that A is par(r_2, −)-regular. Taking $r = \min(r_1, r_2)$, we obtain that A is parallel regular. ∎

3. DIGITIZATION PRESERVING TOPOLOGY

Our definition of a digitization models a real digitization process. Consistent with real sensor output, a digitization is defined with respect to a grid of squares (or cubes in 3D), where each square (or cube) has diameter r. A square (or cube) is a black pixel (or voxel) iff the ratio of the area (volume) of the object to the area (volume) of the field "seen" by the corresponding sensor is greater than some constant threshold value. For any threshold value, we show that the digitization with diameter r of a par(r)-regular set A will be homotopy equivalent to A.

DEFINITION: Let X be any set in the plane (or space). Let $\mathfrak{Q}$ be a cover of the plane (space) by squares (cubes) with diameter r such that the intersection of two squares is empty, a corner point, or an edge (or a face). Such a cover is called a square grid (cubical grid) with diameter r. Each square (cube) in $\mathfrak{Q}$ is either white or black. If we treat the squares of $\mathfrak{Q}$ as points in Z^2 (Z^3) with the corresponding colors, we obtain a digital picture, which will be called a *digitization* of X with diameter r. We will also identify the digitization of X with the union of closed black squares (cubes); i.e., the digitization of X is a closed subset of the plane (or space). Thus, the digitization of X refers to either the digital picture or the union of closed black squares (cubes).

In the following, we define some important digitization classes.

DEFINITION. Let X be any set in the plane (or space). A square (cube) of $p \in \mathfrak{Q}$ is black iff $int(p) \cap X \neq \emptyset$, and white otherwise, where $int(p)$ denotes the interior of p. We will call such a digitization an *intersection digitization* with diameter r of set X. We will denote this digitization

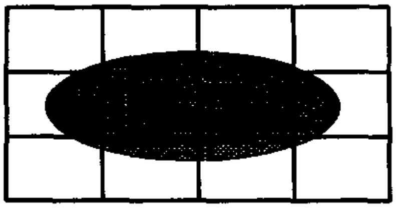

FIG. 5. The union of all squares represents an intersection digitization of an ellipse.

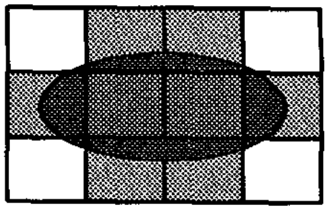

FIG. 7. The gray pixels represent a digitization of an ellipse with the area ratio equal to $\frac{1}{2}$.

by $Dig_\cap(X, r)$, namely $Dig_\cap(X, r) = \bigcup\{p: int(p) \cap X \neq \emptyset\}$. $Dig_\cap(X, r)$ either denotes the digital picture or the union of closed black squares (cubes). See Fig. 5, for example, where the union of all depicted squares represents an intersection digitization of an ellipse.

With respect to real camera digitization, an intersection digitization corresponds to the procedure of coloring a pixel (or voxel) black iff there is part of the object A in the field "seen" by the corresponding sensor.

Now we consider digitization corresponding to the procedure of coloring a pixel (or voxel) black iff the object X fills the whole field seen by the corresponding sensor. For such digitizations, a square p is black iff $p \subseteq X$ and white otherwise. Note that this statement is equivalent to the following rule: a square p is white iff $p \cap X^c \neq \emptyset$ and black otherwise, where X^c denotes the complement of set X. We will refer to such digitization of a set X as *subset digitization* and denote by $Dig_\subset(X, r)$, where $Dig_\subset(X, r) = \bigcup\{p: p \subseteq X\}$. In Fig. 6, the two black squares represent $Dig_\subset(X, r)$, where X is an ellipse.

Next, let us consider digitization corresponding to a procedure in which a pixel is colored black iff the ratio of the area of the object seen by the sensor to the area of the entire field seen by the same sensor is greater than some constant threshold value v. An example is given in Fig. 7, where the gray pixels represent a digitization of the ellipse with the ratio equal to $\frac{1}{2}$.

Let X be any set in the plane (or space). In the 2D case, square $p \in \mathfrak{Q}$ is black iff $area(p \cap X)/area(p) > v$ and white otherwise, and in the 3D case, cube $p \in \mathfrak{Q}$ is black iff $volume\,(p \cap X)/volume(p) > v$ and white otherwise, where $0 \leq v < 1$ is a constant. If we treat the squares of $\mathfrak{Q}$ as points in Z^2 (Z^3) with the corresponding black or white values, we obtain a digital picture, which will be called the *v-digitization* of X with diameter r. We will denote such digitizations by $Dig_v(X, r)$. We recall that we also identify the digitization of X with the union of black closed squares (cubes). Thus, $Dig_v(X, r)$ either denotes the digital picture or the union of black closed squares (cubes). We will also denote $Dig_1(X, r)$ as the digitization in which the ratio of the area (volume) is equal to 1.

We have the following inclusions: $Dig_\subset(X, r) \subseteq Dig_v(X, r) \subseteq Dig_\cap(X, r)$ for every $v \in [0, 1]$ and $Dig_v(X, r) \subseteq Dig_w(X, r)$ if $w \leq v$ for every $v, w \in [0, 1]$. Since our results apply to any of these digitizations, we will hereafter use $Dig(X, r)$ without subscript to denote $Dig_\cap(X, r)$, $Dig_\subset(X, r)$, and $Dig_v(X, r)$ for every $v \in [0, 1]$.

Note that since digitization has been defined by either area or volume, it does not matter whether the squares (cubes) are topologically open or closed or "half-open" and "half-closed." Since it is more convenient to prove the following theorems using closed objects, we stipulate that all black squares (cubes) of cover $\mathfrak{Q}$ are closed.

In the following, we briefly review the concept of homotopy equivalence.

DEFINITION. Let X and Y be two topological spaces. Two functions f, g: $X \to Y$ are said to be *homotopic* if there exists a continuous function H: $X \times [0, 1] \to Y$, where $[0, 1]$ is the unit interval, with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. The function H is called a *homotopy* from f to g. X and Y are called *homotopy equivalent* or of the same *homotopy type* if there exist two functions f: $X \to Y$ and g: $Y \to X$ such that $g \circ f$ is homotopic with the identity over X (id_X) and $f \circ g$ is homotopic with the identity over Y (id_Y).

Intuitively, a homotopy H represents a continuous deformation of the map f to g. As a consequence of the properties of homotopy equivalence, there is a complete correspondence between connected components of X and Y and their complements if X and Y are pathwise connected. The Euler characteristic, as well as the fundamental groups of X and Y, are the same (see [7]). Therefore, we will use homotopy equivalence as a definition for topology preserving.

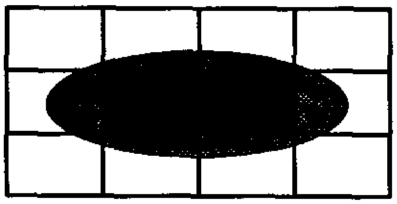

FIG. 6. The two black squares represent a subset digitization of an ellipse.

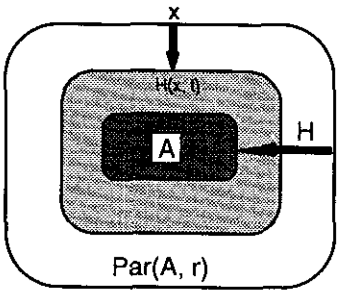

FIG. 8. A par(r, +)-regular set A is a strong deformation retract of $Par(A, r)$.

DEFINITION. We will say that a digitization $Dig(X, r)$ of some set X is *topology preserving* if X and $Dig(X, r)$ are homotopy equivalent.

We now consider a special case of homotopy equivalence called a strong deformation retraction. Intuitively, saying that there is a strong deformation retraction from a set X to a set $Y \subseteq X$ means that we can continuously shrink X to Y.

DEFINITION. Let X and $Y \subseteq X$ be two topological spaces. A continuous function H: $X \times [0, 1] \to X$, where $[0, 1]$ is the unit interval, is called a *strong deformation retraction* of X to Y if $H(x, 1) = x$ and $H(x, 0) = x \in Y$ for every $x \in X$, and $H(x, t) = x$ for every $x \in Y$ and $t \in [0, 1]$. Y is called a *strong deformation retract* of X.

Note that if Y is a strong deformation retract of X, then Y is also homotopy equivalent to X. To see this, take f: $X \to Y$ to be $f(x) = H(x, 0)$ and g: $Y \to X$ to be inclusion.

THEOREM 7. *Let A be a par(r, +)-regular object. Then A is a strong deformation retract of $Par(A, r)$.*

Proof. If $x \in (Par(A, r) - A)$, then there exists a unique normal vector $n(a, r)$, for some $a \in bdA$, such that $x \in n(a, r)$. We define a projection π: $(Par(A, r) - A) \to bdA$ by $\pi(x) = a$, where a is such that $x \in n(a, r)$. From this definition, it follows that $\pi^{-1}(a) = n(a, r)$.

Let H be a function defined as follows (see Fig. 8):

H: $Par(A, r) \times [0, 1] \to Par(A, r)$,

$H(x, t) = x$ for every $x \in A$ and $t \in [0, 1]$,

$H(x, t) = (1 - t)\pi(x) + tx$ for every $x \notin A$ and $t \in [0, 1]$.

Note that $H(x, 1) = x$ for every $x \in Par(A, r)$ and that $H(x, t) = x$ for every $x \in A$ and $t \in [0, 1]$. Note also that $H(x, 0) = \pi(x)$ for all $x \in Par(A, r) - A$. Thus, $H(x, 0) \in A$ for every $x \in Par(A, r)$, i.e., $H(Par(A, r), 0) = A$.

To prove that H is a strong deformation retraction, it remains only to show that H is a continuous function. Clearly, for a fixed x, $H(x, t)$ as a function of t is continuous. If t is fixed, the continuity of $H(x, t)$ as a function of x follows from the continuity of the metric projection π, which implies that if x and y are close to each other, then the line segments $x\pi(x)$ and $y\pi(y)$ are close to each other. Therefore, H is a strong deformation retraction of $Par(A, r)$ to A. ■

As a consequence of this theorem, we obtain the following theorems.

THEOREM 8. *Let A be a par(r)-regular object. Then $Par(A, -r)$ is a strong deformation retract of $Par(A, r)$.*

Proof. Since A is both par(r, +)-regular and par(r, −)-regular and for every $x \in bdPar(A, -r)$, $n(x, 2r)$ is just the union of $-n(y, r)$ and $n(y, r)$ for some $y \in bdA$ (see Fig. 9), we obtain that $n(a, 2r)$ and $n(b, 2r)$ and $n(b, 2r)$ do not intersect for every two distinct points $a, b \in bdPar(A, -r)$. Thus, $Par(A, -r)$ is par(2r, +)-regular. Observe also that $Par(Par(A, -r), 2r) = Par(A, r)$. Applying Theorem 7 to $Par(A, -r)$, we obtain that $Par(A, -r)$ is a strong deformation retract of $Par(A, r)$.

Using the construction in Theorem 7, a strong deformation retraction F of $Par(A, r)$ to $Par(A, -r)$ can also be defined directly:

F: $Par(A, r) \times [0, 1] \to Par(A, r)$

$F(x, t) = x$ for every $x \in Par(A, -r)$ and $t \in [0, 1]$,

$F(x, t) = (1 - t)\pi(x) + tx$
for every $x \in Par(A, r) - Par(A, -r)$ and $t \in [0, 1]$,

where $\pi(x)$ is the only point on $bdPar(A, -r)$ such that $x \in n(\pi(x), 2r)$. ■

THEOREM 9. *Let A be a par(r)-regular set. Then $Par(A, -r)$ is a strong deformation retract of $Dig(A, r)$ for every digitization $Dig(A, r)$.*

Proof. By Theorem 1, $Par(A, r) = Dil(A, r)$. Let p be a closed square or cube with diameter r such that $p \cap A \neq \emptyset$. Since $p \subseteq B(x, r)$ for every closed ball $B(x, r)$ such that $x \in p$, we obtain that $p \subseteq Dil(A, r)$. Therefore, $Dig_{\cap}(A, r) \subseteq Par(A, r)$. For every closed square or cube p with diameter r, it similarly holds that if $p \cap Par(A, -r) \neq \emptyset$, then $p \subseteq A$. Therefore, $Dig_{\cap}(Par(A, -r), r) \subseteq Dig_{\subset}(A, r)$. Since it is clear that $Par(A, -r) \subseteq Dig_{\cap}(Par(A,$

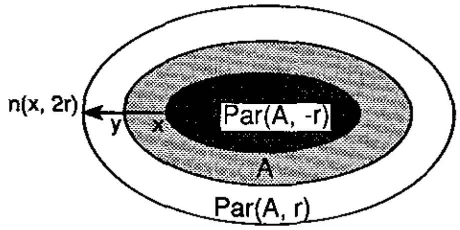

FIG. 9. $Par(A, -r)$ is a strong deformation retract of $Par(A, r)$.

$-r), r)$, we obtain $Par(A, -r) \subseteq Dig_{\subset}(A, r)$. Thus, we obtain the following inclusions: $Par(A, -r) \subseteq Dig_{\subset}(A, r) \subseteq Dig_{\cap}(A, r) \subseteq Par(A, r)$. Since by the definition of $Dig(A, r)$, $Dig_{\subset}(A, r) \subseteq Dig(A, r) \subseteq Dig_{\cap}(A, r)$, we obtain that

$$Par(A, -r) \subseteq Dig(A, r) \subseteq Par(A, r).$$

We construct a strong deformation retraction from $Dig(A, r)$ onto $Par(A, -r)$:

$D: Dig(A, r) \times [0, 1] \to Dig(A, r)$.

If $x \in Par(A, -r)$, then $D(x, t) = x$ for every $t \in [0, 1]$.

In the following, $x \in Dig(A, r) \backslash Par(A, -r)$. Let $p(x)$ be a point on $bdPar(A, -r)$ with the closest distance to x. Let $xp(x)$ be the line segment joining x with $p(x)$. Since $xp(x) \subseteq n(p(x), 2r)$ and $Par(A, -r)$ is par(2r, +)-regular, $p(x)$ is uniquely determined. Therefore, the metric projection p is a continuous function from $Dig(A, r) \backslash Par(A, -r)$ to $bdPar(A, -r)$.

For every line segment $xp(x)$, we define a modified path $mp(x, p(x)) \subset Dig(A, r)$ from x to $p(x)$. If $xp(x) \subset Dig(A, r)$, then $mp(x, p(x)) = xp(x)$.

Now we define $mp(x, p(x))$ in the case where $xp(x) \not\subset Dig(A, r)$. Then there exists a line segment $ab \subset xp(x)$ such that $ab \cap Dig(A, r) = \{a, b\}$. Let $path(a, b)$ be the shortest path from a to b contained in $bdDig(A, r)$ (and thus contained also in $Dig(A, r)$). The existence and uniqueness of such a path will be shown below. If $xa \cup bp(x)$ is contained in $Dig(A, r)$ then $xa \cup path(a, b) \cup bp(x)$ is contained in $Dig(A, r)$, and we define $mp(x, p(x)) = xa \cup path(a, b) \cup bp(x)$. If either xa or $bp(x)$ is not contained in $Dig(A, r)$, then we recursively do this construction for xa or $bp(x)$. We continue this process until the modified path $mp(x, p(x))$ is contained in $Dig(A, r)$. Then we parametrize uniformly $mp(x, p(x))$ with a continuous function f_x: $[0, 1] \to mp(x, p(x))$ such that $f_x(0) = p(x)$ and $f_x(1) = x$.

If $x \in Dig(A, r) \backslash Par(A, -r)$, then we define $D(x, t) = f_x(t)$.

In order to show that D is correctly defined, it remains to show that if there exists a line segment $ab \subset xp(x)$ such that $ab \cap Dig(A, r) = \{a, b\}$, then the shortest path $path(a, b) \subset bdDig(A, r)$ exists and is uniquely determined. This follows from the following fact: Let s be a white square (cube). For every $x \in bdPar(A, -r)$, if $n(x, 2r)$ intersects two different faces of s which are contained in $bdDig(A, r)$, then these two faces share a vertex (an edge) and $n(x, 2r)$ does not contain the diagonal of s.

It is easy to observe that for a fixed x, $D(x, t)$ as a function of t is continuous. If t is fixed, the continuity of $D(x, t)$ as a function of x follows from the continuity of the metric projection p, which implies that if x and y are close to each other, then the line segments $xp(x)$ and $yp(y)$ are close to each other. This again implies that the modified paths $mp(xp(x))$ and $mp(yp(y))$ are close to each other.

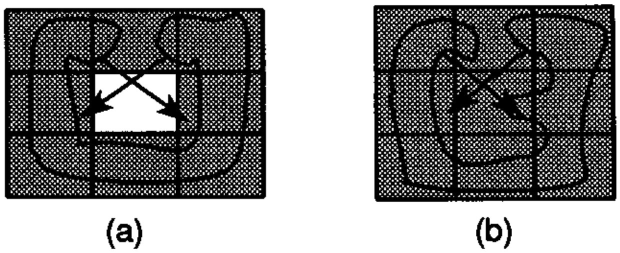

FIG. 10. (a) A and $Dig_{\cap}(A, r)$ are not homotopy equivalent. (b) X and $Dig_{\cap}(X, r)$ are homotopy equivalent.

Finally, we need to establish that D satisfies the other properties of a strong deformation retraction. By definition, $D(x, t) = x$ for every $x \in Par(A, -r)$ and $t \in [0, 1]$. Clearly, $D(x, 1) = x$ and $D(x, 0) \in Par(A, -r)$ for every $x \in Dig(A, r)$. Thus, D is a strong deformation retraction of $Dig(A, r)$ to $Par(A, -r)$. ∎

Now we are ready to prove our main theorems. We recall that $Dig(A, r)$ denotes $Dig_{\cap}(X, r)$, $Dig_{\subset}(X, r)$, and $Dig_v(X, r)$ for every $v \in [0, 1]$, and these digital images model the output of many real digitization processes.

THEOREM 10. *Let A be a par(r)-regular set. Then A and $Dig(A, r)$ are homotopy equivalent.*

Proof. By Theorem 7, A is a strong deformation retract of $Par(A, r)$. By Theorem 8, $Par(A, -r)$ is a strong deformation retract of $Par(A, r)$. Since both sets A and $Par(A, -r)$ are strong deformation retracts of the same set $Par(A, r)$, A and $Par(A, -r)$ are homotopy equivalent (see [2, p. 288]. By Theorem 9, $Par(A, -r)$ and $Dig(A, r)$ are homotopy equivalent. Thus, A and $Dig(A, r)$ are homotopy equivalent. ∎

THEOREM 11. *Let A be a closed subset of the plane or space with a compact boundary such that the second derivative exists at every point $x \in bdA$ and is continuous. Then there always exists a digitization resolution $r > 0$ such that every digitization $Dig(A, r)$ of A is topology preserving.*

Proof. It is a consequence of Theorems 6 and 10. ∎

The assumption that A is a par(r)-regular set is the weakest one which guarantees that the topological structure of the set is preserved by its digitization. We show, for example, that if a set is not par(r)-regular, then its digitization $Dig_{\cap}(A, r)$ can have different topological structure, as illustrated in Fig. 10a, where set A is simple connected, but $Dig_{\cap}(A, r)$ represented by gray squares is not simple connected, since there is a white "hole" in it. Of course, one can always find a set X having some special shape which is not par(r)-regular, yet X and $Dig_{\cap}(X, r)$ are homotopy equivalent, like the set presented in Fig. 10b. Although topology was preserved in digitizing the set shown in Fig.

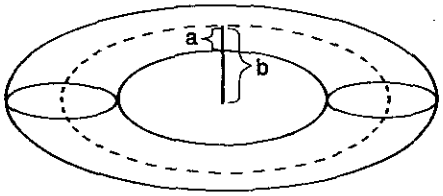

FIG. 11. It is easy to calculate r for a torus T such that each $Dig(T, r)$ is topology preserving.

10b, it is clear that important shape properties were lost. However, as we will show in the next section, if a set A is par(r)-regular, then $Dig_{\cap}(A, r)$ will never significantly change its geometric shape properties.

There are many important object classes used in computer vision and medical imaging that are par(r)-regular for some r and for which the calculation of r is straightforward. One such class is that of planar generalized tubular surfaces, which are constructed by sweeping a planar curve around an axis which is another planar curve. For objects in this class, it is shown in [3] that the parameter curves are also lines of curvature, so that the calculation of r can follow the following scheme. For example, we show how to calculate r for a given torus T such that each $Dig(T, r)$ is topology preserving. By Theorem 11, it is sufficient to find the value of r such that T is par(r)-regular. Using Theorem 6, we know that there exists an $r > 0$ such that T is par(r)-regular. We use the construction given in the proof of Theorem 5 to compute the maximal value of r such that T is par(r)-regular. We assume that T is parameterized as

$$T = (b + a \sin \phi)(\cos \theta)e_1 + (b + a \sin \phi)(\sin \theta)e_2 + (a \cos \phi)e_3,$$

where T is obtained by sweeping a circle of radius a around a circular axis of radius b, where $b > a$, and (e_1, e_2, e_3) form an orthonormal basis in R^3 (Fig. 11).

Following Step 1 of the proof of Theorem 5, we first calculate $k_{\max}$. For a torus $k_{\max}$ is given by $k_{\max} = 1/a$ if $b - a \leq a$, and by $k_{\max} = 1/(b - a)$ if $b - a > a$. Let $r < 1/k_{\max} = \min\{a, b - a\}$. Observe that in the case of a torus, this not only guarantees us that locally in some neighbourhood of every point $x \in T$ the normal vectors of the length r do not pairwise intersect, but also that globally these vectors do not pairwise intersect for all points in T. Thus, if $r < \min\{a, b - a\}$, then $n(x, r) \cap n(y, r) = \emptyset$ and $-n(x, r) \cap -n(y, r) = \emptyset$ for all $x, y \in T$, which means that T is par(r)-regular so that any digitization $Dig(T, r)$ of the torus is topology preserving.

$N_3(P)$	$N_2(P)$	$N_1(P)$
$N_4(P)$	P	$N_0(P)$
$N_5(P)$	$N_6(P)$	$N_7(P)$

FIG. 12. The points in $N_8^*(P)$ are numbered 0 to 7 according to this scheme.

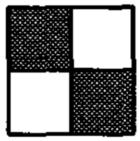

FIG. 13. This pattern and its 90° rotation cannot occur in $Dig_{\cap}(A, r)$.

4. DIGITAL DIFFERENTIAL GEOMETRY

In this section, we show that if A is a par(r)-regular set, then only a few digital patterns can occur as neighborhoods of boundary points in its digitization $Dig_{\cap}(A, r)$. This is very useful for noise detection, since if the neighborhood of a boundary point does not match one of these patterns, it must be due to noise. So, if in a practical application the resolution r of the digitization is such that the parts of the object which must be preserved under the digitization form a par(r)-regular set, then our results allow for efficient noise detection.

We also show that the digitization $Dig_{\cap}(A, r)$ of a par(r)-regular set A will not change the qualitative differential geometric properties of the boundary of A; i.e., a boundary point which is locally convex cannot be digitized to a locally concave pixel and a boundary point which is locally concave cannot be digitized to a locally convex pixel.

First, we need to prove some facts about the local connectedness of par(r)-regular sets.

LEMMA 4. *Let A be par(r, +)-regular. Then $bdA \cap B(x, t)$ is connected for every $t \leq r$ and $x \notin A$. Let A be par(r, −)-regular. Then $bdA \cap B(x, t)$ is connected for every $t \leq r$ and $x \in A - bdA$.*

Proof. We prove only the first part of the theorem.

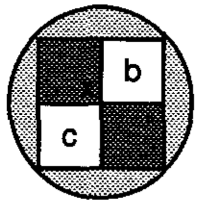

FIG. 14. Let x be the common vertex of all four squares. Then the four squares are contained in $B(x, r)$.

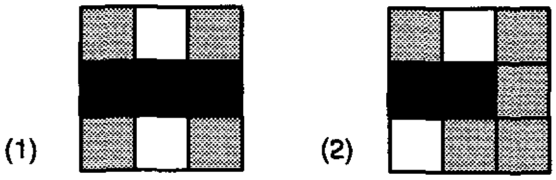

FIG. 15. These patterns and their 90° rotations cannot occur in $Dig_{\cap}(A, r)$, where the light gray points can be either black or white.

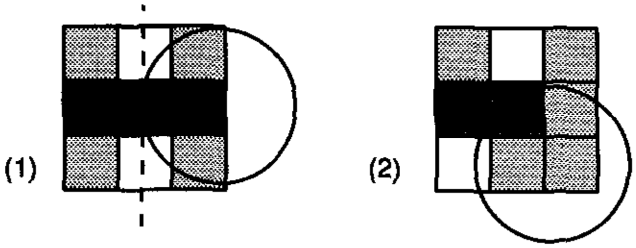

FIG. 16. These patterns and their 90° rotations cannot occur in $Dig_{\cap}(A, r)$. The circles illustrate $iob(x, r)$.

The proof of the second part is analogous. We show that $bdA \cap B(x, t)$ is connected for every $t \leq r$ and $x \notin A$. Let this not be the case; i.e., there exist $t \leq r$ and two components C and D of $bdA \cap B(x, t)$ for some $x \notin A$. Since A is par(r, +)-regular, A is also par(t, +)-regular. Applying Lemma 3 to C and D separately, we obtain that there exist $c \in C \cap B(x, t)$ such that $x \in n(c, t)$ and $d \in D \cap B(x, t)$ such that $x \in n(d, t)$. Thus, $n(c, t) \cap n(d, t) \neq \emptyset$. This is an inconsistency, since $n(c, t)$ and $n(d, t)$ are normal vectors at distinct points $c, d \in bdA$. Thus, $bdA \cap B(x, t)$ is connected for every $t \leq r$ and $x \notin A$. ■

LEMMA 5. *Let A be par$(r, +)$-regular. Then $A \cap B(x, t)$ is connected for every $t \leq r$ and $x \notin A$. Let A be par$(r, -)$-regular. Then $A \cap B(x, t)$ is connected for every $t \leq r$ and $x \in A - bdA$.*

Proof. We prove only the first part of the theorem. The proof of the second part is analogous. Let $t \leq r$. If $A \cap B(x, t)$ were disconnected, then $bdA \cap B(x, t) \neq \emptyset$ and $bdA \cap B(x, t)$ would be disconnected, which is impossible by Lemma 4. ■

THEOREM 12. *Let A be par(r)-regular. Then $A \cap B(x, t)$ is connected for every $t \leq r$ and $x \in R^2$ (or $x \in R^3$).*

Proof. By Lemma 5, it remains to consider the case in which $x \in bdA$. Assume that there exist two components C and D in $A \cap B(x, t)$ for some $t \leq r$. Then x belongs to one of them, say C. By Lemma 3, applied to D and $x \in D$, we obtain that there exists $s \in D$ such that $x \in n(s, t)$. However, then $x \in n(x, t) \cap n(s, t)$. Thus, $A \cap B(x, t)$ must be connected. ■

Now we review some definitions from digital topology which are based on [5, 9]. As usual in digital topology, we assume that all sets are subsets of $\mathbb{Z}^2$ or $\mathbb{Z}^3$. The points in a set S will be termed *black* or *foreground* points, while those of the complement S^c will be termed *white* or *background* points.

The *4-neighbors* (or *direct neighbors*) of a point (x, y) in $\mathbb{Z}^2$ are its four horizontal and vertical neighbors $(x + 1, y)$, $(x - 1, y)$ and $(x, y + 1)$ $(x, y - 1)$. The *8-neighbors* of a point (x, y) in $\mathbb{Z}^2$ are its four horizontal and vertical neighbors together with its four diagonal neighbors $(x + 1, y + 1)$, $(x + 1, y - 1)$ and $(x - 1, y + 1)$, $(x - 1, y - 1)$. The *n-neighborhood* of a point P is the set $N_n(P)$ consisting of P and its n-neighbors, where $n = 4$ or 8. $N_n^*(P)$ is the set of all neighbors of P without P itself, i.e., $N_n^*(P) = N_n(P) - \{P\}$. In the 2D case, $N_8(p)$ is also referred to as $N(p)$ and called the *neighborhood* of p. The points in $N_8^*(P)$ are numbered 0 to 7 according to the scheme in Fig. 12.

DEFINITION. In the 2D case, a black point p is a *boundary point* if one of its 4-neighbors is white.

THEOREM 13 (2*D case*). *Let A be par$(r, +)$-regular. Then the pattern shown in Fig. 13 and its 90° rotation cannot occur in $Dig_{\cap}(A, r)$.*

Proof. Let x be the common vertex of all four squares and let the two closed white squares be denoted as in Fig. 14. Note first that $x \notin A$, since otherwise all four squares would be black. Then $A \cap B(x, r)$ is disconnected, since $B(x, r) - (b \cup c)$ is disconnected and $(A \cap B(x, r)) \subseteq (B(x, r) - (b \cap c))$. This is however impossible according to Lemma 5. ■

By this theorem $Dig_{\cap}(A, r)$ is well composed; i.e., every 8-component of black points, as well as of white points, is

FIG. 17. *All possible configurations modulo reflection and rotation which can occur on the boundary of the digitization $Dig_{\cap}(A, r)$ of a par(r)-regular set A.*

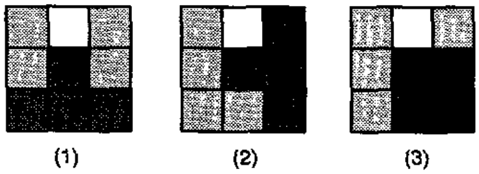

FIG. 18. $iob(x, r)$ must intersect at least all black squares in one of these configurations.

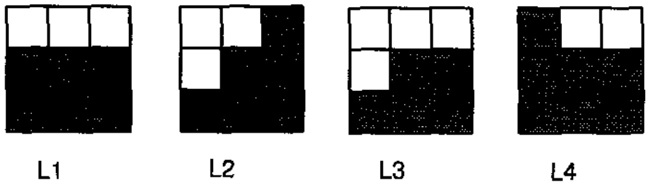

FIG. 19. Linear digital neighborhoods.

also a 4-component. Well-composed sets have very nice digital topological properties [6], in particular, the Jordan Curve Theorem holds and the Euler characteristic is locally computable. These results imply that many algorithms for digital image processing can be simpler and faster.

THEOREM 14. *(2D case) Let A be par(r)-regular. Then the patterns shown in Fig.* 15 *and their* 90° *rotations cannot occur in* $Dig_{\cap}(A, r)$, *where the light gray points can be either black or white.*

Proof. Assume that pattern (1) in Fig. 15 occurs in $Dig_{\cap}(A, r)$. We denote the middle square by p. From Theorem 12, it follows that $A \cap (p \cup N_0(p))$ and $A \cap (p \cup N_4(p))$ are connected, and therefore $A \cap (p \cup N_0(p) \cup N_4(p))$ is connected. Let $a \in A \cap N_0(p)$ and $b \in A \cap N_4(p)$. Then there is a path joining a with b which goes through p. Therefore, there exists a point $x \in bdA \cap p \cap L$, where l is a straight line parallel to $p \cap N_0(p)$ which goes through the center of p. (the dashed line in Fig. 16). Yet $iob(x, r) \subset A$, but this implies that either $N_2(p)$ or $N_6(p)$ is black, since $iob(x, r)$ must intersect at least one of these closed squares. The obtained inconsistency implies that pattern (1) cannot occur in $Dig_{\cap}(A, r)$.

Assume now that pattern (2) in Fig. 15 occurs in $Dig_{\cap}(A, r)$. We denote the middle square by p. From Theorem 12, it follows that $A \cap (p \cup N_4(p))$ is connected. Let $a \in A \cap p$ and $b \in A \cap N_4(p)$. Then there is a path joining a with b which intersects line segment $p \cap N_4(p)$. Therefore, there exists a point $x \in bdA \cap p \cap N_4(p)$. Yet $iob(x, r) \subseteq A$, but this implies that either $N_2(p)$ or $N_5(p)$ is black, since $iob(x, r)$ must intersect at least one of these closed squares. The obtained inconsistency implies that pattern (2) cannot occur in $Dig_{\cap}(A, r)$. ∎

The following theorem is a very important result for recovering differential geometric properties of an object and for noise detection. If a template occurs that is different from those enumerated in the theorem, we can be sure that it must be due to noise.

THEOREM 15 *(2D case). Let A be par(r)-regular. Then the neighborhood N(p) of a (4-)boundary black point* $p \in Dig_{\cap}(A, r)$ *can have only one of the configurations presented in Fig.* 17 *(modulo reflection and* 90° *rotation).*

Proof. One of the 4-neighbors of p must be white, say $N_2(p)$, since p is a (4-)boundary point. Since, $N_2(p)$ is white, there exists $x \in bdA \cap p$, because otherwise, $p \subseteq A$, and $N_2(p)$ would be black. By Theorem 3, $iob(x, r) \subseteq A$. Since the radius of $iob(x, r)$ is equal to the diameter of a single square in $Dig_{\cap}(A, r)$, $iob(x, r)$ must intersect at least all black squares in one of the configurations shown in Fig. 18 (modulo reflection and rotation), where the light gray points can be of either color. By Theorems 13 and 14, it is easy to check that only the configurations listed in the theorem can extend configurations (1)–(3). ∎

Observe that the set of realizable patterns given in Theorem 15 is very small. Moreover, this set is minimal; i.e., the number of patterns which can occur as the neighborhood of a (4-)boundary black point in $Dig_{\cap}(A, r)$ cannot be further reduced, since it is easy to construct a par(r)-regular set A such that $Dig_{\cap}(A, r)$ contains each of the patterns in Fig. 17. Knowing that these seven configurations constitute all possible configurations (modulo reflection and rotation) which can occur on the boundary of the digitization $Dig_{\cap}(A, r)$ of a par(r)-regular set A, it is now a simple task to classify each of these configurations with respect to their differential geometric properties. First, we identify all possible boundary configurations that are digitizations of a half-plane.

THEOREM 16 *(2D case). Let H be a closed half-plate. Then p is a boundary point of* $Dig_{\cap}(H, r)$ *iff the neighborhood of p has one of the patterns shown in Fig.* 19.

Proof. It is enough to observe that (i) each of the four patterns can be a legal intersection digitization of some closed half plane, and (ii) the remaining three configurations in Fig. 17 can only occur in $Dig_{\cap}(A, r)$ if either A or A^c is not convex. ∎

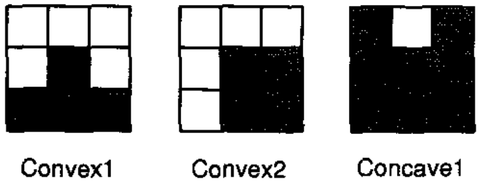

FIG. 20. Convex and concave digital neighborhoods.

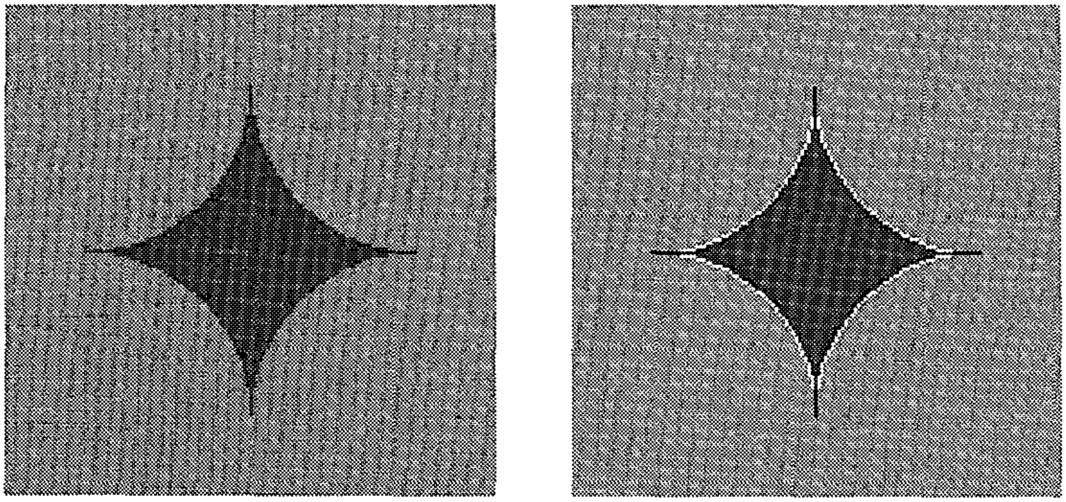

FIG. 21. For the superelliptic object (left), the boundary points having legal neighborhood configurations are colored white, while those having illegal neighborhood configurations are colored black (right).

DEFINITION. Based on Theorem 16, we define a *linear digital neighborhood* of p as any neighborhood configuration $N(p)$ corresponding to L1, L2, L3, L4 in Fig. 19 modulo reflection and 90° rotation. Next we define a *convex digital neighborhood* of p as any neighborhood configuration $N(p)$ corresponding to Convex1 or Convex2 in Fig. 20 modulo reflection and 90° rotation. We define a *concave digital neighborhood* of p as any neighborhood configuration $N(p)$ corresponding to Concave1 in Fig. 20 modulo reflection and 90° rotation.

THEOREM 17. *Let x be a boundary point of a planar set A such that $A \cap B(x, 2r)$ is convex, and let p be the square of $Dig_\cap(A, r)$ containing x. If p is a boundary point of $Dig_\cap(A, r)$, then $N(p)$ cannot be digitally concave.*

Proof. Note that $N(p)$ is contained in $B(x, 2r)$. If $N(p)$ were Concave1, then there would exist $b \in N_1(p) \cap A$ and $c \in N_3(p) \cap A$. Since $a, b \in A \cap B(x, 2r)$ and $A \cap B(x, 2r)$ is convex, line segment bc must also be contained in A, and therefore $A \cap N_2(p) \neq \emptyset$. Thus, $N_2(p)$ would be black.

THEOREM 18. *Let x be a boundary point of a planar set A such that $A \cap B(x, 2r)$ is concave, and let p be the square of $Dig_\cap(A, r)$ containing point x. If p is a boundary point of $Dig_\cap(A, r)$, then $N(p)$ cannot be digitally convex.*

Proof. Note that $N(p)$ is contained in $B(x, 2r)$. If $N(p)$ were Convex1, then $N_0(p)$, $N_4(p) \subset A^c$. Since $A^c \cap B(x, 2r)$ is convex, the convex hull CH of $N_0(p) \cup N_4(p)$ is contained in A^c. Since p is contained in CH, it is also contained in A^c. However, then p would be white. This shows that $N(p)$ cannot be Convex1.

If $N(p)$ were Convex2, then $N_1(p)$, $N_5(p) \subset A^c$. Since $A^c \cap B(x, 2r)$ is convex, the convex hull CH of $N_1(p) \cap N_5(p)$ is contained in A^c. Since p is contained in CH, it would be white. Thus, $N(p)$ cannot be Convex2.

Our results, although stated for parallel regular objects, are also helpful to analyze the shape properties of nonparallel regular objects, as the following example illustrates. We use the legal neighborhood configurations shown in Fig. 17 for corner detection. Assume that the object is piecewise par(r)-regular and we want to find the corners of the object. Note that good candidates for corners are sections on the object boundary where the par(r)-regular pieces are joined together. Fig. 21(left) shows the intersection digitization of a set bounded by a superelliptic curve. In Fig. 21(right), the boundary points having legal neighborhood configurations are colored white, while those having illegal neighborhood configurations are colored black. Thus, the black-colored boundary points correspond to sections of the object boundary that are not par(r)-regular. These points correctly identify the "corners" of the superelliptic curve.

5. CONCLUSIONS AND FUTURE WORK

In this paper, we proved that topological and differential geometric properties are preserved under digitization. For a par(r)-regular set A in the plane and 3D space, digitizations $Dig_\cap(A, r)$, $Dig_\subset(A, r)$, and $Dig_v(A, r)$ are topology preserving for every $0 \leq v \leq 1$. This result is important for practical applications, since these digitizations model the output of many real digitization processes, and for a large class of real objects, including medical objects, a constant r can always be computed such that the object is par(r)-regular. Observe that our results can also be applied to multicolor digital images. A multicolor digital picture can be reduced to a binary picture if we temporarily concentrate on one color c and treat all points of color c as foreground and all other points as background. Using this reduction, all results for binary images stated in this paper will also hold for multicolor images.

Only a few digital patterns can occur as neighborhoods of a boundary point in $Dig_{\cap}(A, r)$ of a par(r)-regular 2D set A. This is very useful for noise detection, since if the neighborhood of a boundary point does not match one of these patterns, it must be due to noise. Note that this result also applies to digitizations of 3D objects, since digitizing a 3D object can be modeled as the digitization of its 2D projections. Moreover, the digitization $Dig_{\cap}(A, r)$ of a par(r)-regular 2D set A does not change the qualitative differential geometric properties of the boundary of A.

Although our results hold for parallel regular sets, they can also be applied to recover shape properties of nonparallel regular sets as the example in the last section illustrates. Future work includes generalizing the digitization model to handle piecewise par(r)-regular sets.

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