Markov Random Fields and Stochastic Image Models

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   (d) Parameter estimation
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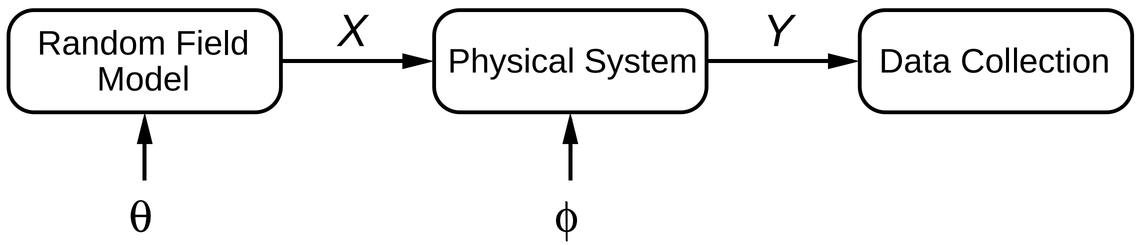
The Bayesian Approach

θ - Random field model parameters

X - Unknown image

ϕ - Physical system model parameters

Y - Observed data

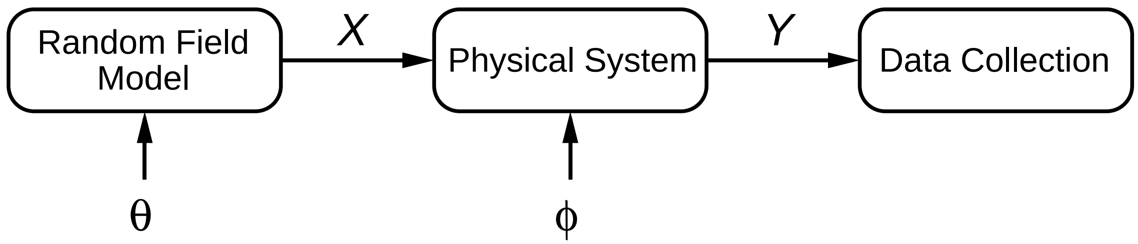


- Random field may model:
  - Achromatic/color/multispectral image
  - Image of discrete pixel classifications
  - Model of object cross-section
- Physical system may model:
  - Optics of image scanner
  - Spectral reflectivity of ground covers (remote sensing)
  - Tomographic data collection

Bayesian Versus Frequentist?

- How does the Bayesian approach differ?
  - Bayesian makes assumptions about prior behavior.
  - Bayesian requires that you choose a model.
  - A **good** prior model can improve accuracy.
  - But model mismatch can impair accuracy
- When should you use the frequentist approach?
  - When (# of data samples)>>(# of unknowns).
  - When an accurate prior model does not exist.
  - When prior model is not needed.
- When should you use the Bayesian approach?
  - When (# of data samples)≈(# of unknowns).
  - When model mismatch is tolerable.
  - When accuracy without prior is poor.

Examples of Bayesian Versus Frequentist?

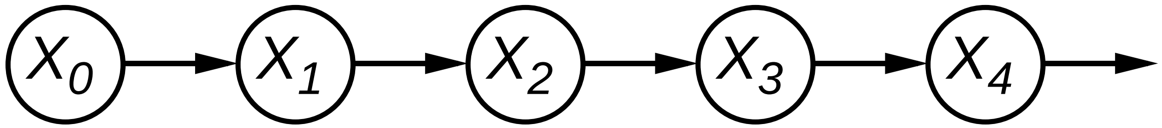


- Bayesian model of image X
  - (# of image points)$\approx$(# of data points.)
  - Images have unique behaviors which may be modeled.
  - Maximum likelihood estimation works poorly.
  - Reduce model mismatch by estimating parameter θ.
- Frequentist model for θ and ϕ
  - (# of model parameters)$<<$(# of data points.)
  - Parameters are difficult to model.
  - Maximum likelihood estimation works well.

Markov Chains

- Topics to be covered:
  - 1-D properties
  - Parameter estimation
  - 2-D Markov Chains
- **Notation:** Upper case $\Rightarrow$ Random variable

Markov Chains



- Definition of (homogeneous) Markov chains

$$p(x_n|x_i\ i<n) = p(x_n|x_{n-1})$$

- Therefore, we may show that the probability of a sequence is given by

$$p(x) = p(x_0)\prod_{n=1}^{N} p(x_n|x_{n-1})$$

- Notice: X_n is **not** independent of X_{n+1}

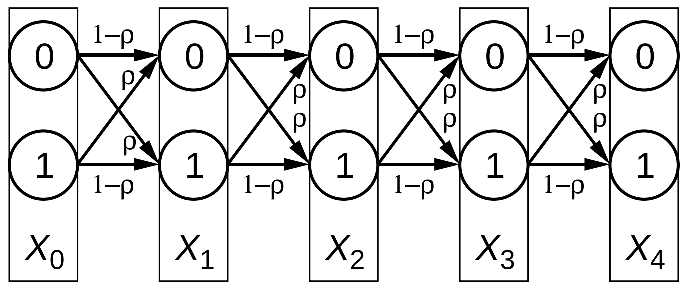
$$p(x_n|x_i\ i\neq n) = p(x_n|x_{n-1}, x_{n+1})$$

Parameters of Markov Chain

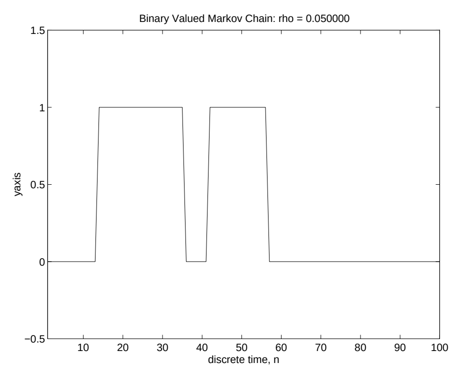
- Transition parameters are:

$$\theta_{j,i} = p(x_n = i | x_{n-1} = j)$$

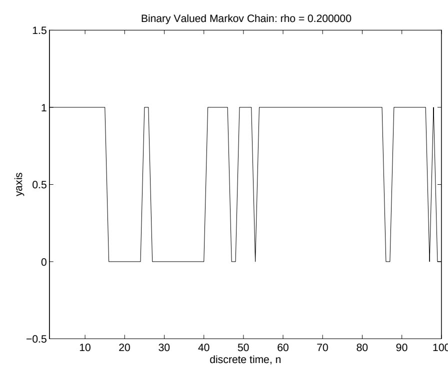
- Example: $\theta = \begin{bmatrix} 1-\rho & \rho \\ \rho & 1-\rho \end{bmatrix}$



- ρ is the probability of changing state.



$\rho = 0.05$



$\rho = 0.2$

Parameter Estimation for Markov Chains

- Maximum likelihood (ML) parameter estimation

$$\hat{\theta} = \arg\max_{\theta} p(x|\theta)$$

- For Markov chain

$$\hat{\theta}_{j,i} = \frac{h_{j,i}}{\sum_k h_{j,k}}$$

where $h_{j,i}$ is the histogram of transitions

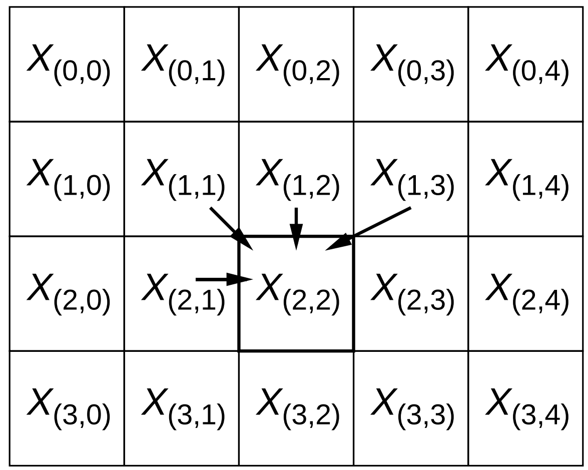
$$h_{j,i} = \sum_n \delta(x_n = i \;\&\; x_{n-1} = j)$$

- Example

$$x_n = 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1$$

$$\theta = \begin{bmatrix} h_{0,0} & h_{0,1} \\ h_{1,0} & h_{1,1} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 6 \end{bmatrix}$$

2-D Markov Chains



- Advantages:
  - Simple expressions for probability
  - Simple parameter estimation
- Disadvantages:
  - No natural ordering of pixels in image
  - Anisotropic model behavior

Discrete State Markov Random Fields

- Topics to be covered:
  - Definitions and theorems
  - 1-D MRF's
  - Ising model
  - M-Level model
  - Line process model

Markov Random Fields

- Noncausal model
- Advantages of MRF's
  - Isotropic behavior
  - Only local dependencies
- Disadvantages of MRF's
  - Computing probability is difficult
  - Parameter estimation is difficult
- Key theoretical result: Hammersley-Clifford theorem

Definition of Neighborhood System and Clique

- Define

  S - set of lattice points

  s - a lattice point, $s \in S$

  X_s - the value of X at s

  ∂s - the neighboring points of s

- A neighborhood system ∂s must be symmetric

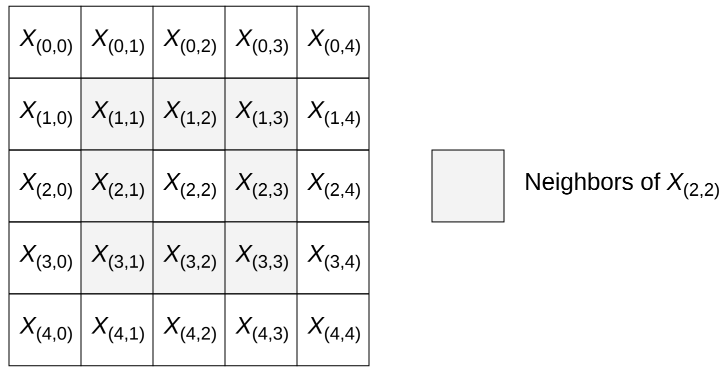
$$r \in \partial s \Rightarrow s \in \partial r \text{ also } s \notin \partial s$$

- A clique is a set of points, c, which are all neighbors of each other

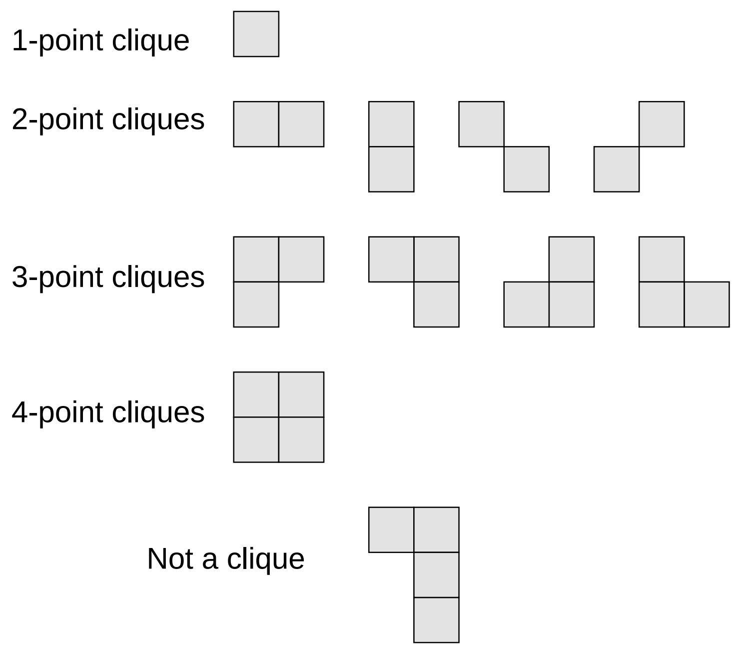
$$\forall s, r \in c, r \in \partial s$$

Example of Neighborhood System and Clique

- Example of 8 point neighborhood



- Example of cliques for 8 point neighborhood



Gibbs Distribution

x_c - The value of X at the points in clique c.

$V_c(x_c)$ - A potential function is any function of x_c.

- A (discrete) density is a Gibbs distribution if

$$p(x) = \frac{1}{Z} \exp \left\{ - \sum_{c \in \mathcal{C}} V_c(x_c) \right\}$$

  $\mathcal{C}$ is the set of all cliques

  Z is the normalizing constant for the density.

- Z is known as the **partition function**.
- $U(x) = \sum_{c \in \mathcal{C}} V_c(x_c)$ is known as the **energy function**.

Markov Random Field

- Definition: A random object X on the lattice S with neighborhood system ∂s is said to be a Markov random field if for all $s \in S$

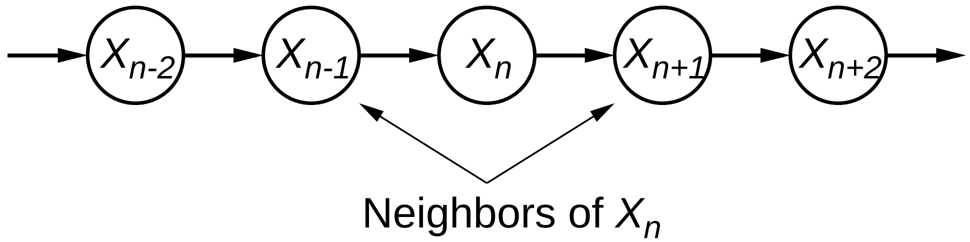
$$p(x_s | x_r \text{ for } r \neq s) = p(x_s | x_{\partial r})$$

Hammersley-Clifford Theorem[14]

$$\left(\begin{array}{c} X \text{ is a Markov random field} \\ \& \\ \forall x,\ P\{X = x\} > 0 \end{array} \right) \Longleftrightarrow \left(\begin{array}{c} P\{X = x\} \text{ has the form} \\ \text{of a Gibbs distribution} \end{array} \right)$$

- Gives you a method for writing the density for a MRF
- Does not give the value of Z, the partition function.
- Positivity, $P\{X = x\} > 0$, is a technical condition which we will generally assume.

Markov Chains are MRF's



- Neighbors of n are $\partial n = \{n-1, n+1\}$
- Cliques have the form $c = \{n-1, n\}$
- Density has the form

$$\begin{aligned} p(x) &= p(x_0) \prod_{n=1}^{N} p(x_n | x_{n-1}) \\ &= p(x_0) \exp\left\{ \sum_{n=1}^{N} \log p(x_n | x_{n-1}) \right\} \end{aligned}$$

- The potential functions have the form

$$V(x_n, x_{n-1}) = \log p(x_n | x_{n-1})$$

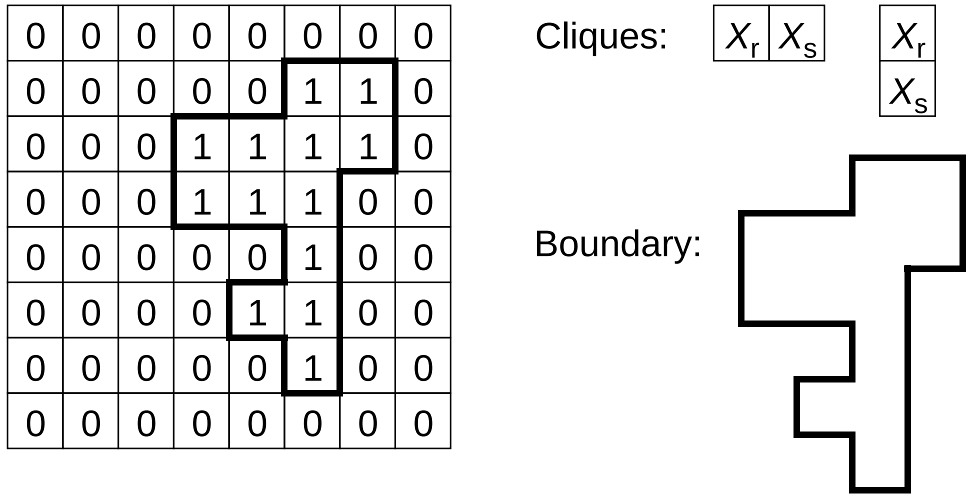
1-D MRF's are Markov Chains

- Let X_n be a 1-D MRF with $\partial n = \{n-1, n+1\}$
- The discrete density has the form of a Gibbs distribution

$$p(x) = p(x_0) \exp\left\{ \sum_{n=1}^{N} V(x_n, x_{n-1}) \right\}$$

- It may be shown that this is a Markov Chain.
- Transition probabilities may be difficult to compute.

The Ising Model: A 2-D MRF[100]



- Potential functions are given by

$$V(x_r, x_s) = \beta\delta(x_r \neq x_s)$$

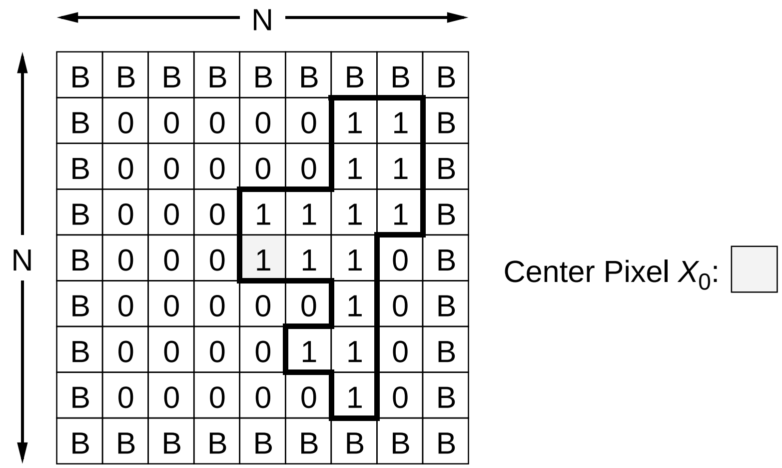
where β is a model parameter.

- Energy function is given by

$$\sum_{c \in \mathcal{C}} V_c(x_c) = \beta(\text{Boundary length})$$

- Longer boundaries $\Rightarrow$ less probable

Critical Temperature Behavior[127, 126, 100]



- $\frac{1}{\beta}$ is analogous to temperature.
- Peierls showed that for $\beta > \beta_c$

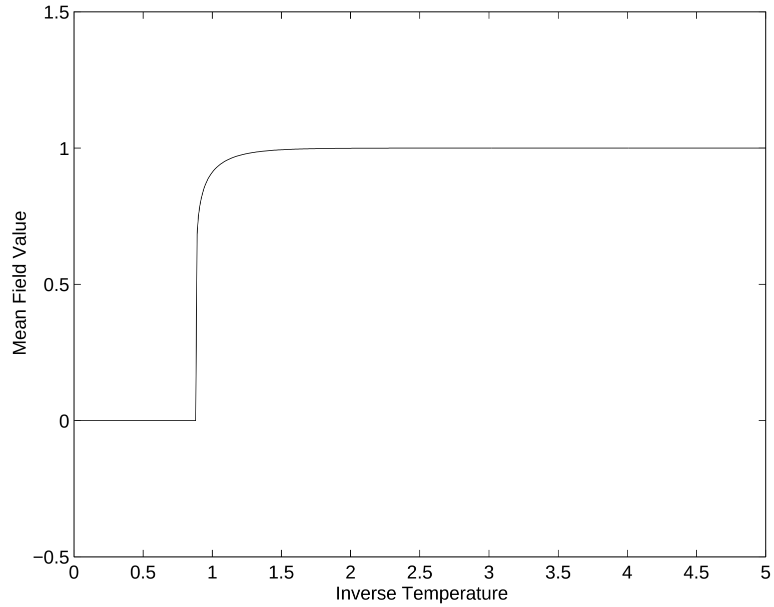
$$\lim_{N \to \infty} P(X_0 = 0 | B = 0) \neq \lim_{N \to \infty} P(X_0 = 0 | B = 1)$$

- The effect of the boundary does not diminish as $N \to \infty$!
- $\beta_c \approx .88$ is known as the critical temperature.

Critical Temperature Analysis[122]

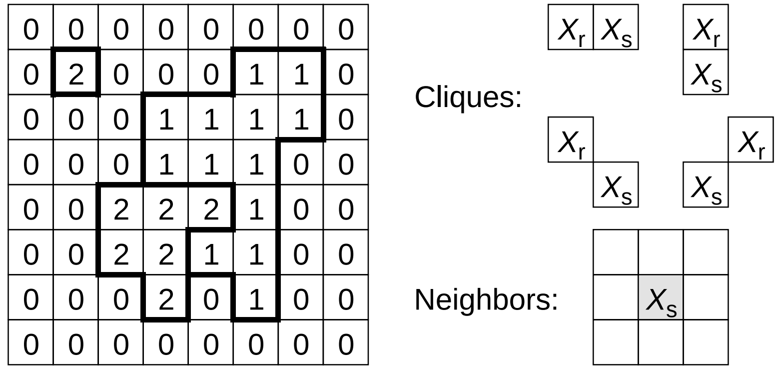
- Amazingly, Onsager was able to compute

$$E[X_0|B=1] = \begin{cases} \left(1 - \frac{1}{(\sinh(\beta))^4}\right)^{1/8} & \text{if } \beta > \beta_c \\ 0 & \text{if } \beta < \beta_c \end{cases}$$



- Onsager also computed an analytic expression for $Z(T)$!

M-Level MRF[16]



- Define $\mathcal{C}_1 \triangleq$ (hor./vert. cliques) and $\mathcal{C}_2 \triangleq$ (diag. cliques)
- Then

$$V(x_r, x_s) = \begin{cases} \beta_1 \delta(x_r \neq x_s) & \text{for } \{x_r, x_s\} \in \mathcal{C}_1 \\ \beta_2 \delta(x_r \neq x_s) & \text{for } \{x_r, x_s\} \in \mathcal{C}_2 \end{cases}$$

- Define

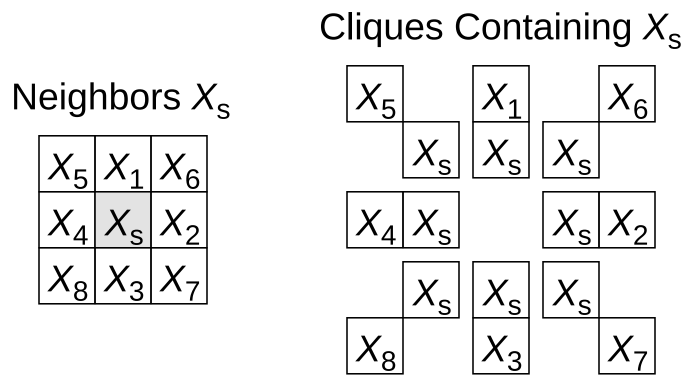
$$t_1(x) \triangleq \sum_{\{s,r\} \in \mathcal{C}_1} \delta(x_r \neq x_s)$$

$$t_2(x) \triangleq \sum_{\{s,r\} \in \mathcal{C}_2} \delta(x_r \neq x_s)$$

- Then the probability is given by

$$p(x) = \frac{1}{Z} \exp\left\{-(\beta_1 t_1(x) + \beta_2 t_2(x))\right\}$$

Conditional Probability of a Pixel



- The probability of a pixel given all other pixels is

$$p(x_s|x_{i\neq s}) = \frac{\frac{1}{Z}\exp\left\{-\Sigma_{c\in\mathcal{C}} V_c(x_c)\right\}}{\Sigma_{x_s=0}^{M-1}\frac{1}{Z}\exp\left\{-\Sigma_{c\in\mathcal{C}} V_c(x_c)\right\}}$$

- Notice: Any term $V_c(x_c)$ which does not include x_s cancels.

$$p(x_s|x_{i\neq s}) = \frac{\exp\left\{-\beta_1\Sigma_{i=1}^{4}\delta(x_s\neq x_i) - \beta_2\Sigma_{i=5}^{8}\delta(x_s\neq x_i)\right\}}{\Sigma_{x_s=0}^{M-1}\exp\left\{-\beta_1\Sigma_{i=1}^{4}\delta(x_s\neq x_i) - \beta_2\Sigma_{i=5}^{8}\delta(x_s\neq x_i)\right\}}$$

Conditional Probability of a Pixel (Continued)

Neighbors X_s

1	1	0
1	X_s	0
0	0	0

$V_1(0,x_{\partial s}) = 2 \quad V_2(0,x_{\partial s}) = 1$

$V_1(1,x_{\partial s}) = 2 \quad V_2(1,x_{\partial s}) = 3$

- Define

$$v_1(x_s, \partial x_s) \triangleq \# \text{ of horz./vert. neighbors} \neq x_s$$
$$v_2(x_s, \partial x_s) \triangleq \# \text{ of diag. neighbors} \neq x_s$$

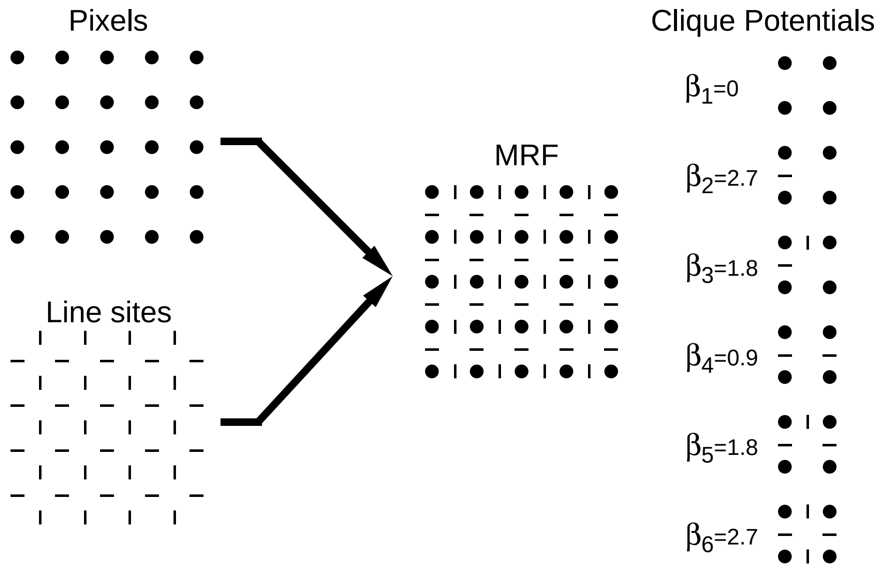
- Then

$$p(x_s|x_{i\neq s}) = \frac{1}{Z'} \exp\left\{-\beta_1 v_1(x_s, \partial x_s) - \beta_2 v_2(x_s, \partial x_s)\right\}$$

where Z' is an easily computed normalizing constant

- When $\beta_1, \beta_2 > 0$, X_s is most likely to be the majority neighboring class.

Line Process MRF [68]



- Line sites fall between pixels
- The values $\beta_1, \cdots, \beta_2$ determine the potential of line sites
- The potential of pixel values is

$$V(x_s, x_r, l_{r,s}) = \begin{cases} (x_s - x_r)^2 & \text{if } l_{r,s} = 0 \\ 0 & \text{if } l_{r,s} = 1 \end{cases}$$

- The field is
  - Smooth between line sites
  - Discontinuous at line sites

Simulation

- Topics to be covered:
  - Metropolis sampler
  - Gibbs sampler
  - Generalized Metropolis sampler

Generating Samples from a Gibbs Distribution

- How do we generate a random variable X with a Gibbs distribution?

$$p(x) = \frac{1}{Z} \exp\left\{-U(x)\right\}$$

- Generally, this problem is difficult.
- Markov Chains can be generated sequentially
- Non-causal structure of MRF's makes simulation difficult.

The Metropolis Sampler[118, 100]

- How do we generate a sample from a Gibbs distribution?

$$p(x) = \frac{1}{Z} \exp\left\{-U(x)\right\}$$

- Start with the sample x^k, and generate a new sample W with probability $q(w|x^k)$.

  Note: $q(w|x^k)$ must be symmetric.

$$q(w|x^k) = q(x^k|w)$$

- Compute $\Delta E(W) = U(W) - U(x^k)$, then do the following:

  If $\Delta E(W) < 0$

  – Accept: $X^{k+1} = W$

  If $\Delta E(W) \geq 0$

  – Accept: $X^{k+1} = W$ with probability $\exp\{-\Delta E(W)\}$
  – Reject: $X^{k+1} = x^k$ with probability $1 - \exp\{-\Delta E(W)\}$

Ergodic Behavior of Metropolis Sampler

- The sequence of random fields, X^k, form a Markov chain.
- Let $p(x^{k+1}|x^k)$ be the transition probabilities of the Markov chain.
- Then X^k is reversible

$$p(x^{k+1}|x^k)\exp\{-U(x^k)\} = \exp\{-U(x^{k+1})\}p(x^k|x^{k+1})$$

- Therefore, if the Markov chain is irreducible, then

$$\lim_{k\to\infty} P\{X^k = x\} = \frac{1}{Z}\exp\{-U(x)\}$$

- If every state can be reached, then as $k \to \infty$, X^k will be a sample from the Gibbs distribution.

Example Metropolis Sampler for Ising Model

	0	
1	x_s	0
	0	

- Assume $x_s^k = 0$.
- Generate a binary R.V., W, such that $P\{W = 0\} = 0.5$.

$$\begin{aligned} \Delta E(W) &= U(W) - U(x_s^k) \\ &= \begin{cases} 0 & \text{if } W = 0 \\ 2\beta & \text{if } W = 1 \end{cases} \end{aligned}$$

If $\Delta E(W) < 0$

– Accept $X_s^{k+1} = W$

If $\Delta E(W) \geq 0$

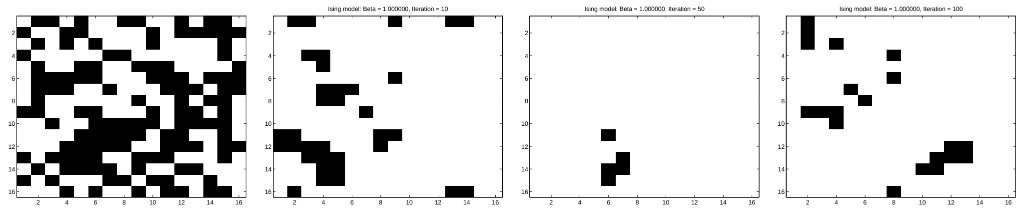
– Accept: $X_s^{k+1} = W$ with probability $\exp\{-\Delta E(W)\}$

– Reject: $X_s^{k+1} = x_s^k$ with probability $1 - \exp\{-\Delta E(W)\}$

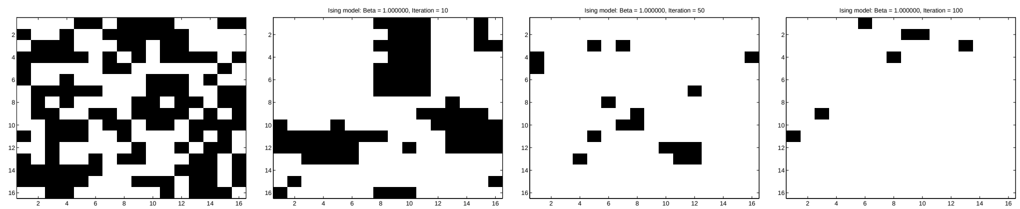
- Repeat this procedure for each pixel.
- **Warning:** for $\beta > \beta_c$ convergence can be extremely slow!

Example Simulation for Ising Model($\beta = 1.0$)

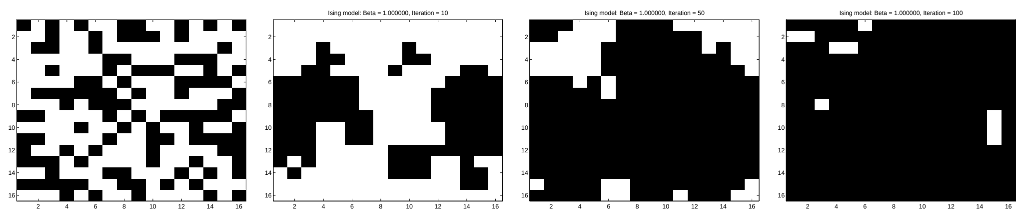
- Test 1



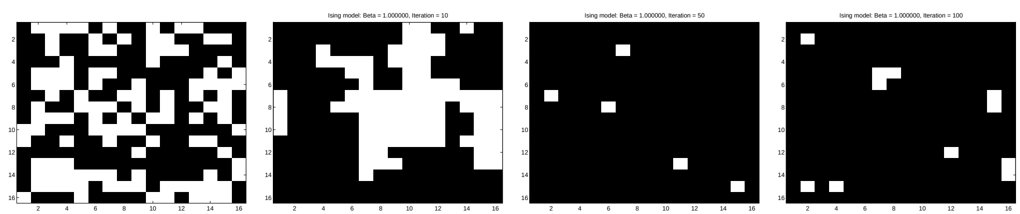
- Test 2



- Test 3



- Test 3



Advantages and Disadvantages of Metropolis Sampler

- Advantages
  - Can be implemented whenever ΔE is easy to compute.
  - Has guaranteed geometric convergence.
- Disadvantages
  - Can be slow if there are many rejections.
  - Is constrained to use a symmetric transition function $q(x^{k+1}|x^k)$.

Gibbs Sampler[68]

- Replace each point with a sample from its conditional distribution

$$p(x_s | x_i^k \; i \neq s) = p(x_s | x_{\partial s})$$

- Scan through all the points in the image.
- Advantage
  - Eliminates need for rejections $\Rightarrow$ faster convergence
- Disadvantage
  - Generating samples from $p(x_s | x_{\partial s})$ can be difficult.

Generalized Metropolis Sampler[80, 129]

- Hastings and Peskun generalized the Metropolis sampler for transition functions $q(w|x^k)$ which are not symmetric.
- The acceptance probability is then

$$\alpha(x_s^k, w) = \min\left\{1, \frac{q(x^k|w)}{q(w|x^k)} \exp\{-\Delta E(w)\}\right\}$$

- Special cases

$$q(w|x^k) = q(x^k|z) \Rightarrow \text{conventional Metropolis}$$
$$q(w_s|x^k) = p(x_s^k|x_{\partial s}^k)\Big|_{x_s^k=w_s} \Rightarrow \text{Gibbs sampler}$$

- Advantage
  - Transition function may be chosen to minimize rejections[76]

Parameter Estimation for Discrete State MRF's

- Topics to be covered:
  - Why is it difficult?
  - Coding/maximum pseudolikehood
  - Least squares

Why is Parameter Estimation Difficult?

- Consider the ML estimate of β for an Ising model.
- Remember that

$$t_1(x) = (\#\text{ horz. and vert. neighbors of different value.})$$

- Then the ML estimate of β is

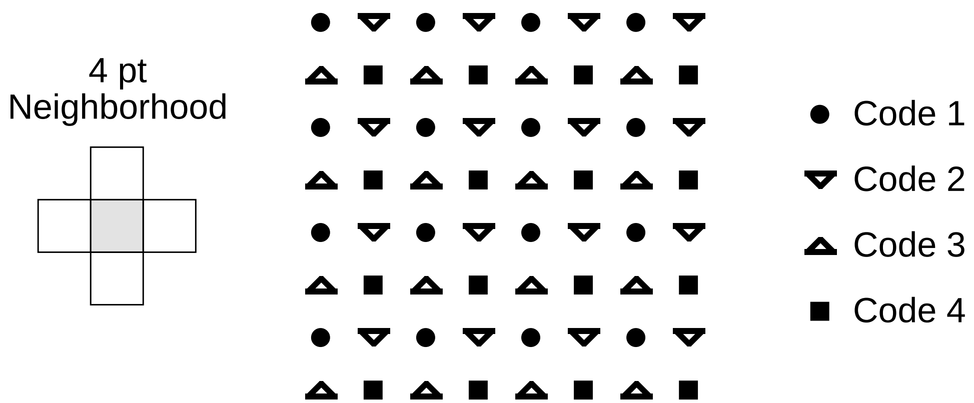
$$\begin{aligned}\hat{\beta} &= \arg\max_{\beta} \left\{ \frac{1}{Z(\beta)} \exp\left\{-\beta t_1(x)\right\} \right\} \\ &= \arg\max_{\beta} \left\{-\beta t_1(x) - \log Z(\beta)\right\}\end{aligned}$$

- However, $\log Z(\beta)$ has an intractable form

$$\log Z(\beta) = \log \sum_{x} \exp\left\{-\beta t_1(x)\right\}$$

- Partition function can not be computed.

Coding Method/Maximum Pseudolikelihood[15, 16]



- Assume a 4 point neighborhood
- Separate points into four groups or codes.
- Group (code) contains points which are conditionally independent given the other groups (codes).

$$\hat{\beta} = \arg\max_{\beta} \prod_{s \in \text{Code}_k} p(x_s | x_{\partial s})$$

- This is tractable (but not necessarily easy) to compute

Least Squares Parameter Estimation[49]

- It can be shown that for an Ising model

$$\log \frac{P\{X_s = 1|x_{\partial s}\}}{P\{X_s = 0|x_{\partial s}\}} = -\beta \left(V_1(1|x_{\partial s}) - V_1(0|x_{\partial s}) \right)$$

- For each unique set of neighboring pixel values, $x_{\partial s}$, we may compute
  - The observed rate of $\log \frac{P\{X_s=1|x_{\partial s}\}}{P\{X_s=0|x_{\partial s}\}}$
  - The value of $(V_1(1|x_{\partial s}) - V_1(0|x_{\partial s}))$
  - This produces a set of over-determined linear equations which can be solved for β.
- This least squares method is easily implemented.

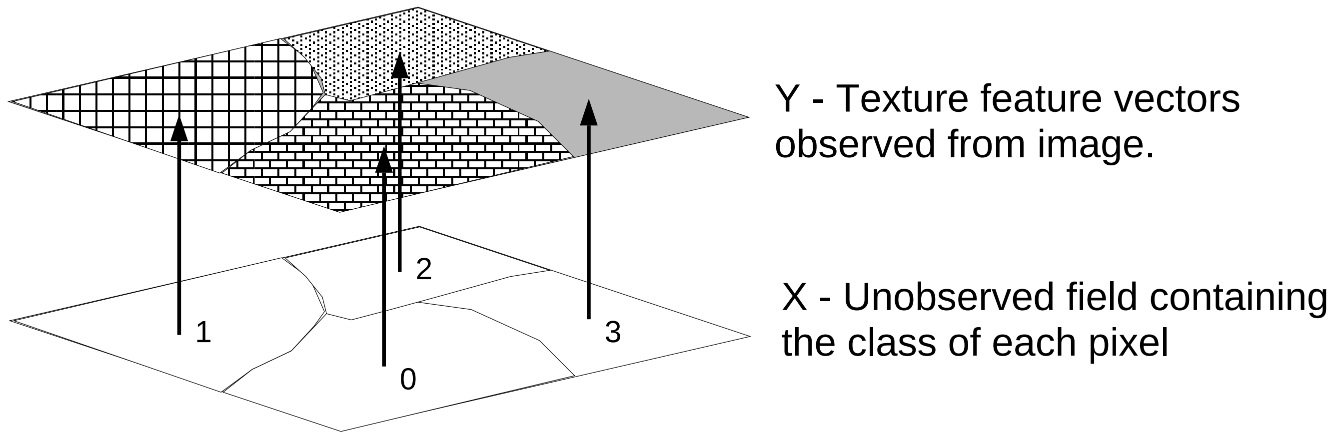
Theoretical Results in Parameter Estimation for MRF's

- Inconsistency of ML estimate for Ising model[130, 131]
  - Caused by critical temperature behavior.
  - Single sample of Ising model cannot distinguish between high β with mean $1/2$, and low β with large mean.
  - Not identifiable
- Consistency of maximum pseudolikelihood estimate[69]
  - Requires an identifiable parameterization.

Application of MRF's to Segmentation

- Topics to be covered:
  - The Model
  - Bayesian Estimation
  - MAP Optimization
  - Parameter Estimation
  - Other Approaches

Bayesian Segmentation Model



- Discrete MRF is used to model the segmentation field.
- Each class is represented by a value $X_s \in \{0, \cdots, M-1\}$
- The joint probability of the data and segmentation is

$$P\{Y \in dy, X = x\} = p(y|x)p(x)$$

where

– $p(y|x)$ is the data model

– $p(x)$ is the segmentation model

Bayes Estimation

- $C(x, X)$ is the cost of guessing x when X is the correct answer.
- $\hat{X}$ is the estimated value of X.
- $E[C(\hat{X}, X)]$ is the expected cost (risk).

- Objective: Choose the estimator $\hat{X}$ which minimizes $E[C(\hat{X}, X)]$.

Maximum *A Posteriori* (MAP) Estimation

- Let $C(x, X) = \delta(x \neq X)$
- Then the optimum estimator is given by

$$
\begin{aligned}
\hat{X}_{MAP} &= \arg\max_x p_{x|y}(x|Y) \\
&= \arg\max_x \log \frac{p_{y,x}(Y,x)}{p_y(Y)} \\
&= \arg\max_x \{\log p(Y|x) + \log p(x)\}
\end{aligned}
$$

- Advantage:
  - Can be computed through direct optimization
- Disadvantage:
  - Cost function is unreasonable for many applications

Maximizer of the Posterior Marginals (MPM) Estimation[116]

- Let $C(x, X) = \sum_{s \in S} \delta(x_s \neq X_s)$
- Then the optimum estimator is given by

$$\hat{X}_{MPM} = \arg\max_{x_s} p_{x_s|Y}(x_s|Y)$$

- Compute the most likely class for each pixel
- Method:
  - Use simulation method to generate samples from $p_{x|y}(x|y)$.
  - For each pixel, choose the most frequent class.

- Advantage:
  - Minimizes number of misclassified pixels
- Disadvantage:
  - Difficult to compute

MAP Optimization for Segmentation

- Assume the data model

$$p_{y|x}(y|x) = \prod_{s \in S} p(y_s|x_s)$$

- And the prior model (Ising model)

$$p_x(x) = \frac{1}{Z'} \exp\{-\beta t_1(x)\}$$

- Then the MAP estimate has the form

$$\hat{x} = \arg\min_x \left\{-\log p_{y|x}(y|x) + \beta t_1(x)\right\}$$

- This optimization problem is very difficult

Iterated Conditional Modes [16]

- The problem:

$$\hat{x}_{MAP} = \arg\min_x \left\{ -\sum_{s\in S} \log p_{y_s|x_s}(y_s|x_s) + \beta t_1(x) \right\}$$

- Iteratively minimize the function with respect to each pixel, x_s.

$$\hat{x}_s = \arg\min_{x_s} \left\{ -\log p_{y_s|x_s}(y_s|x_s) + \beta v_1(x_s|x_{\partial s}) \right\}$$

- This converges to a local minimum in the cost function

Simulated Annealing [68]

- Consider the Gibbs distribution

$$\frac{1}{Z}\exp\left\{-\frac{1}{T}U(x)\right\}$$

where

$$U(x) = \sum_{s\in S} \log p_{y_s|x_s}(y_s|x_s) + \beta t_1(x)$$

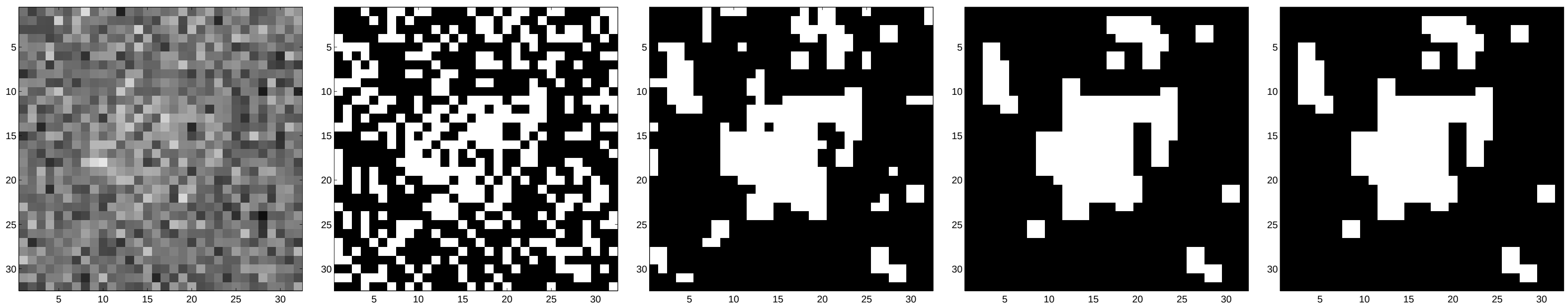
- As $T \rightarrow 0$, the distribution becomes clustered about $\hat{x}_{MAP}$.
- Use simulation method to generate samples from distribution.
- Slowly let $T \rightarrow 0$.
- If $T_k = \frac{T_1}{1+\log k}$ for iteration k, the the simulation converges to $\hat{x}_{MAP}$ almost surely.
- Problem: This is very slow!

Multiscale MAP Segmentation

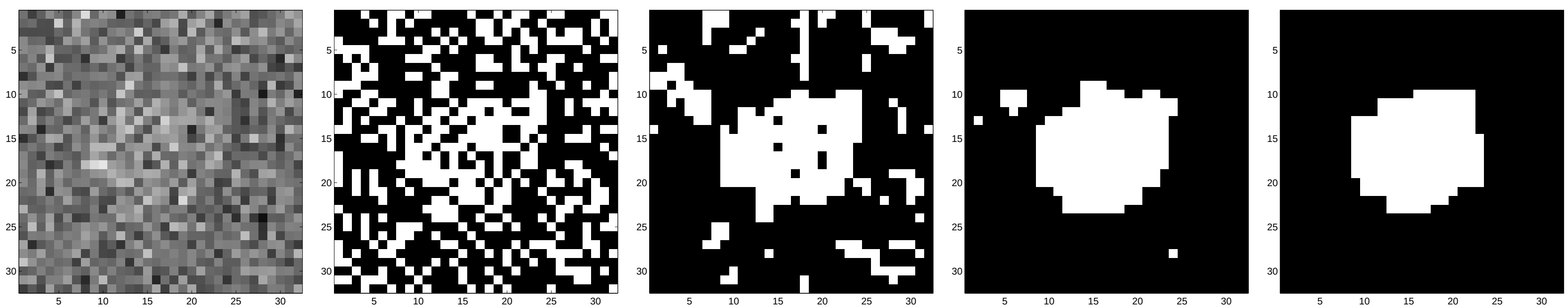
- Renormalization theory[72]
  - Theoretically results in the exact MAP segmentation
  - Requires the computation of intractable functions
  - Can be implemented with approximation
- Multiscale resolution segmentation[23]
  - Performs ICM segmentation in a coarse-to-fine sequence
  - Each MAP optimization is initialized with the solution from the previous coarser resolution
  - Used the fact that a discrete MRF constrained to be block constant is still a MRF.
- Multiscale Markov random fields[97]
  - Extended MRF to the third dimension of scale
  - Formulated a parallel computational approach

Segmentation Example

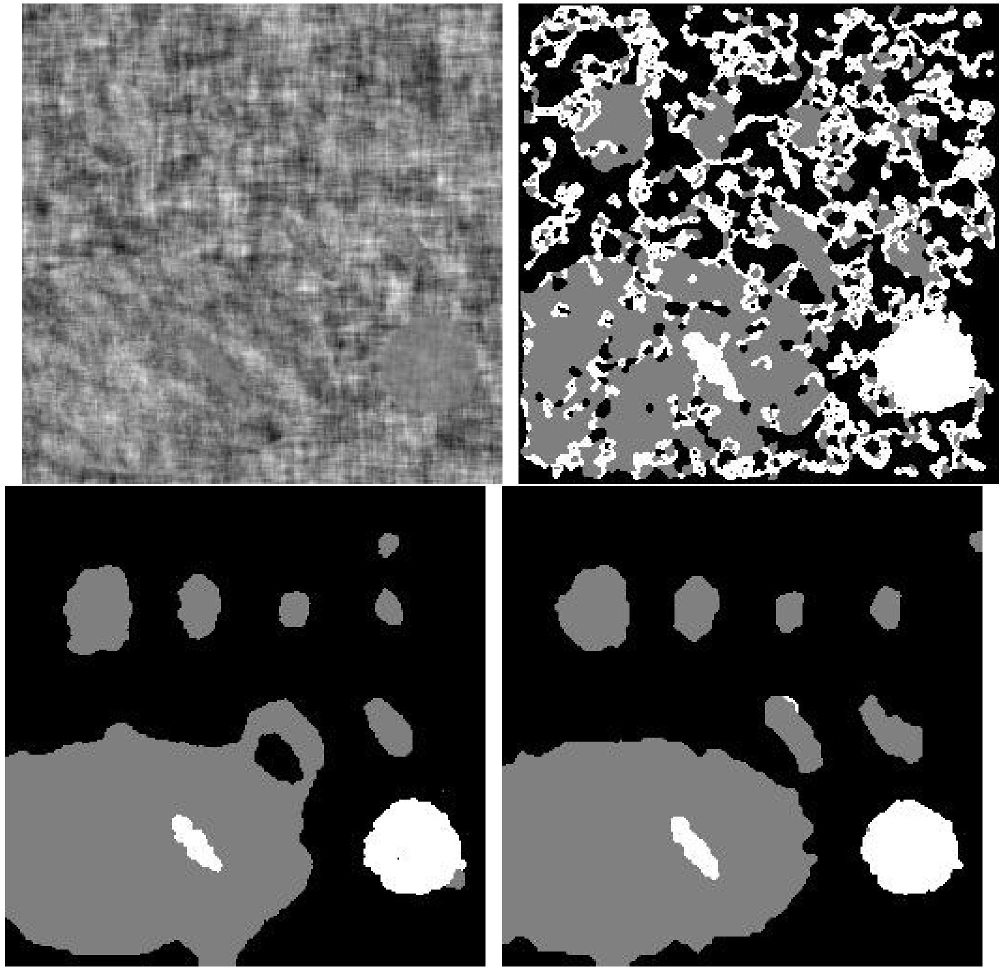
- Iterated Conditional Modes (ICM): ML ; ICM 1; ICM 5; ICM 10



- Simulated Annealing (SA): ML ; SA 1; SA 5; SA 10



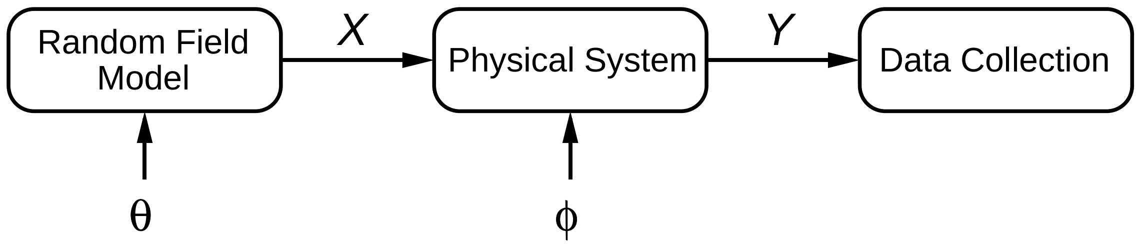
Texture Segmentation Example



a	b
c	d

a) Synthetic image with 3 textures b) ICM - 29 iterations c) Simulated Annealing - 100 iterations d) Multiresolution - 7.8 iterations

Parameter Estimation



- Question: How do we estimate θ from Y?
- Problem: We don't know X!
- Solution 1: Joint MAP estimation [104]

$$(\hat{\theta}, \hat{x}) = \arg\max_{\theta,x} p(y, x|\theta)$$

  – Problem: The solution is biased.

- Solution 2: Expectation maximization algorithm [9, 70]

$$\hat{\theta}^{k+1} = \arg\max_{\theta} E[\log p(Y, X|\theta)|Y = y, \theta^k]$$

  – Expectation may be computed using simulation techniques or mean field theory.

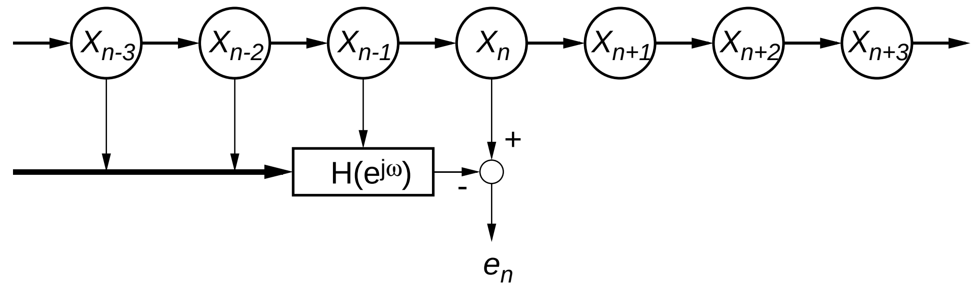
Other Approaches to using Discrete MRFs

- Dynamic programming does not work in 2-D, but a number of researchers have formulated approximate recursive solutions to MAP estimation[48, 169].
- Mean field theory has also been studied as a method for computing the MPM estimate[176].

Gaussian Random Process Models

- Topics to be covered:
  - Autoregressive (AR) models
  - Simultaneous Autoregressive (SAR) models
  - Gaussian MRF's
  - Generalization to 2-D

Autoregressive (AR) Models



$$e_n = x_n - \sum_{k=1}^{\infty} x_{n-k} h_k$$

- $H(e^{j\omega})$ is an optimal predictor $\Rightarrow e(n)$ is white noise.
- The density for the N point vector X is given by

$$p_x(x) = \frac{1}{Z} \exp\left\{ -\frac{1}{2} x^t \mathbf{A}^t \mathbf{A} x \right\}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & & -h_{m-n} \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

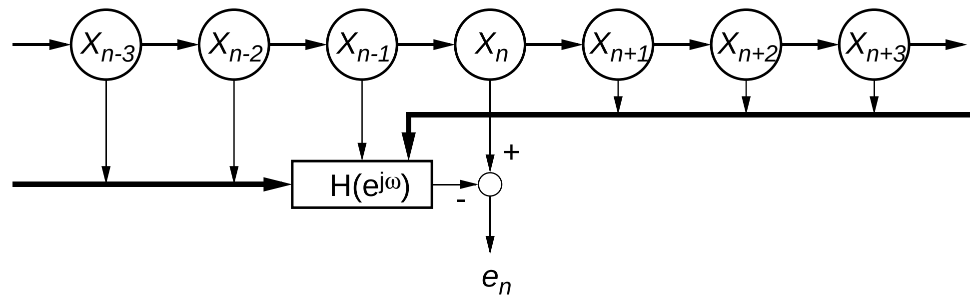
$$Z = (2\pi)^{N/2} |\mathbf{A}|^{-1} = (2\pi)^{N/2}$$

- The power spectrum of X is

$$S_x(e^{j\omega}) = \frac{\sigma_e^2}{|1 - H(e^{j\omega})|^2}$$

Simultaneous Autoregressive (SAR) Models[95, 94]

$$e_n = x_n - \sum_{k=1}^{\infty} (x_{n-k} - x_{n+k}) h_k$$



- $e(n)$ is white noise $\Rightarrow H(e^{j\omega})$ is **not** an optimal non-causal predictor.
- The density for the N point vector X is given by

$$p_x(x) = \frac{1}{Z} \exp\left\{ -\frac{1}{2} x^t \mathbf{A}^t \mathbf{A} x \right\}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & & -h_{m-n} \\ & \ddots & \\ -h_{n-m} & & 1 \end{bmatrix}$$

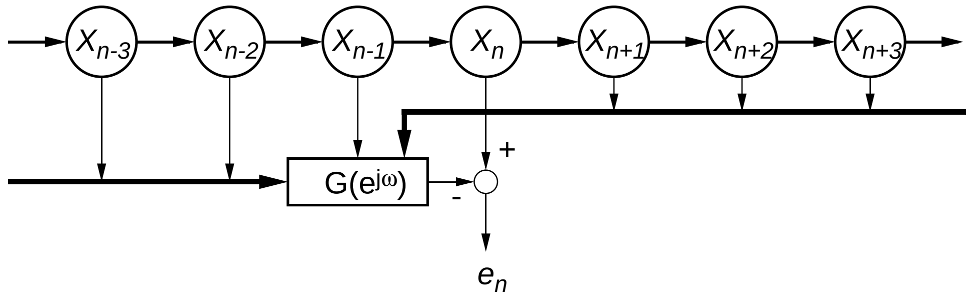
$$Z = (2\pi)^{N/2} |\mathbf{A}|^{-1} \approx (2\pi)^{N/2} \exp\left\{ -\frac{N}{2\pi} \int_{-\pi}^{\pi} \log |1 - H(e^{j\omega})| d\omega \right\}$$

- The power spectrum of X is

$$S_x(e^{j\omega}) = \frac{\sigma_e^2}{|1 - H(e^{j\omega})|^2}$$

Conditional Markov (CM) Models (i.e. MRF's)[95, 94]

$$e_n = x_n - \sum_{k=1}^{\infty}(x_{n-k} - x_{n+k})g_k$$



- $G(e^{j\omega})$ is an optimal non-causal predictor $\Rightarrow$ $e(n)$ is **not** white noise.
- The density for the N point vector X is given by

$$p_x(x) = \frac{1}{Z}\exp\left\{-\frac{1}{2}x^t\mathbf{B}x\right\}$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & & -g_{m-n} \\ & \ddots & \\ -g_{n-m} & & 1 \end{bmatrix}$$

$$Z = (2\pi)^{N/2}|\mathbf{B}|^{-1/2} \approx (2\pi)^{N/2}\exp\left\{-\frac{N}{4\pi}\int_{-\pi}^{\pi}\log(1 - G(e^{j\omega}))d\omega\right\}$$

- The power spectrum of X is

$$S_x(e^{j\omega}) = \frac{\sigma_e^2}{1 - G(e^{j\omega})}$$

Generalization to 2-D

- Same basic properties hold.
- Circulant matrices become circulant block circulant.
- Toeplitz matrices become Toeplitz block Toeplitz.
- SAR and MRF models are more important in 2-D.

Non-Gaussian Continuous State MRF's

- Topics to be covered:
  - Quadratic functions
  - Non-Convex functions
  - Continuous MAP estimation
  - Convex functions

Why use Non-Gaussian MRF's?

- Gaussian MRF's do not model edges well.
- In applications such as image restoration and tomography, Gaussian MRF's either
  - Blur edges
  - Leave excessive amounts of noise

Gaussian MRF's

- Gaussian MRF's have density functions with the form

$$p(x) = \frac{1}{Z} \exp\left\{ -\sum_{s\in S} a_s x_s^2 - \sum_{\{s,r\}\in C} b_{sr}|x_s - x_r|^2 \right\}$$

- We will assume $a_s = 0$.
- The terms $|x_s - x_r|^2$ penalize rapid changes in gray level.
- MAP estimate has the form

$$\hat{x} = \arg\min_x \left\{ -\log p(y|x) + \sum_{\{s,r\}\in C} b_{sr}|x_s - x_r|^2 \right\}$$

- **Problem:** Quadratic function, $|\cdot|^2$, excessively penalizes image edges.

Non-Gaussian MRF's Based on Pair-Wise Cliques

- We will consider MRF's with pair-wise cliques

$$p(x) = \frac{1}{Z} \exp\left\{ - \sum_{\{s,r\}\in C} b_{sr} \rho\left(\frac{x_s - x_r}{\sigma}\right)\right\}$$

$|x_s - x_r|$ - is the change in gray level.

σ - controls the gray level variation or scale.

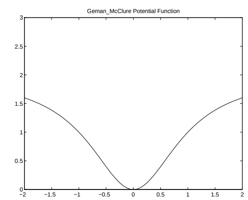
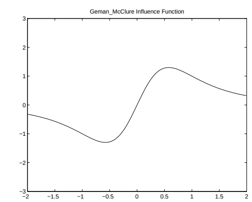
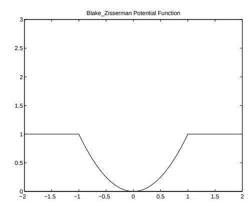
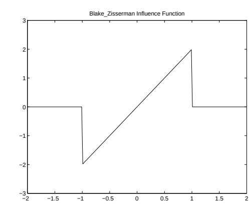
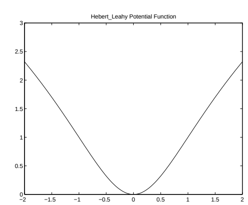
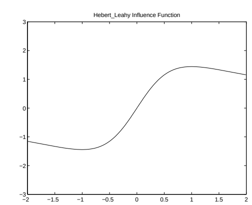
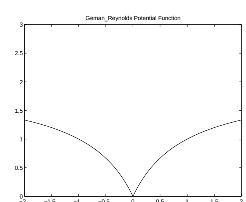
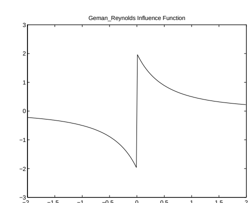
$\rho(\Delta)$:

- Known as the potential function.
- Determines the cost of abrupt changes in gray level.
- $\rho(\Delta) = |\Delta|^2$ is the Gaussian model.

$\rho'(\Delta) = \frac{d\rho(\Delta)}{d\Delta}$:

- Known as the influence function from "M-estimation"[139, 85].
- Determines the attraction of a pixel to neighboring gray levels.

Non-Convex Potential Functions

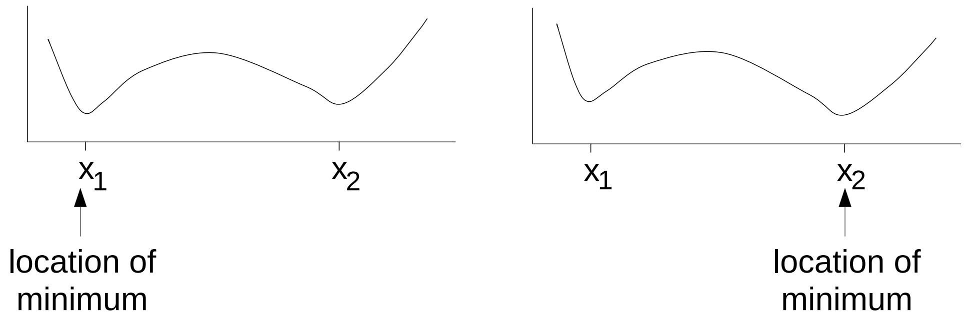
Authors	$\rho(\Delta)$	Ref.	Potential func.	Influence func.
Geman and McClure	$\frac{\Delta^2}{1+\Delta^2}$	[70, 71]		
Blake and Zisserman	$\min\left\{\Delta^2, 1\right\}$	[20, 19]		
Hebert and Leahy	$\log\left(1+\Delta^2\right)$	[81]		
Geman and Reynolds	$\frac{\|\Delta\|}{1+\|\Delta\|}$	[66]		

Properties of Non-Convex Potential Functions

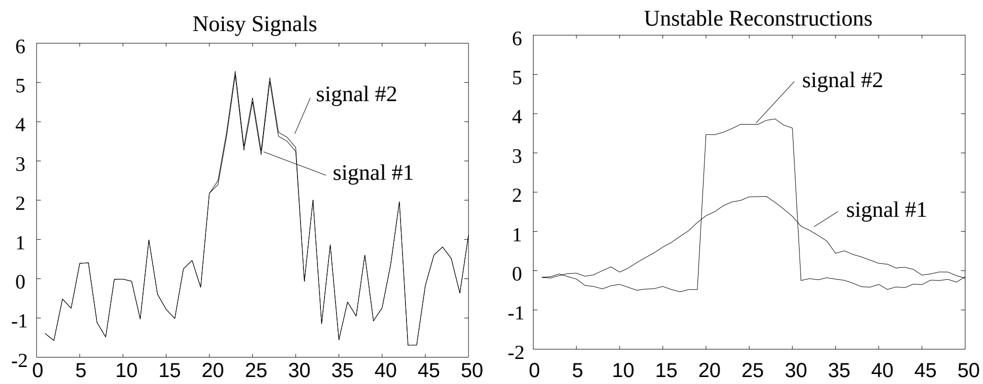
- Advantages
  - Very sharp edges
  - Very general class of potential functions
- Disadvantages
  - Difficult (impossible) to compute MAP estimate
  - Usually requires the choice of an edge threshold
  - **MAP estimate is a discontinuous function of the data**

Continuous (Stable) MAP Estimation[25]

- Minimum of non-convex function can change abruptly.

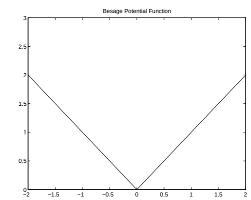
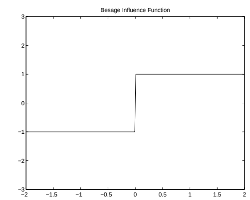
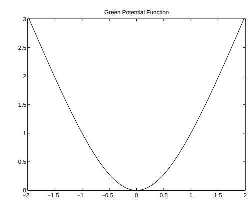
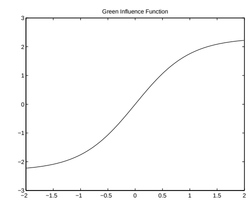
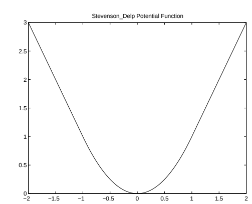
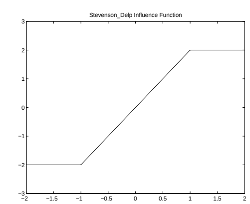
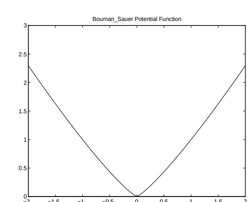
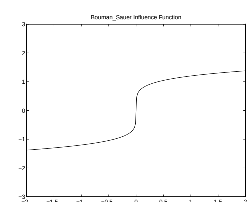


- Discontinuous MAP estimate for Blake and Zisserman potential.



- Theorem:[25] - If the log of the posterior density is **strictly convex**, then the MAP estimate is a continuous function of the data.

Convex Potential Functions

Authors(Name)	$\rho(\Delta)$	Ref.	Potential func.	Influence func.
Besag	$\|\Delta\|$	[17]		
Green	$\log\cosh\Delta$	[75]		
Stevenson and Delp (Huber function)	$\min\{\|\Delta\|^2, 2\|\Delta\|-1\}$	[155]		
Bouman and Sauer (Generalized Gaussian MRF)	$\|\Delta\|^p$	[25]		

Properties of Convex Potential Functions

- Both $\log\cosh(\Delta)$ and Huber functions
  - Quadratic for $|\Delta| << 1$
  - Linear for $|\Delta| >> 1$
  - Transition from quadratic to linear determines edge threshold.
- Generalized Gaussian MRF (GGMRF) functions
  - Include $|\Delta|$ function
  - Do not require an edge threshold parameter.
  - Convex and differentable for $p > 1$.

Parameter Estimation for Continuous MRF's

- Topics to be covered:
  - Estimation of scale parameter, σ
  - Estimation of temperature, T, and shape, p

ML Estimation of Scale Parameter, σ, for Continuous MRF's [26]

- For any continuous state Gibbs distribution

$$p(x) = \frac{1}{Z(\sigma)} \exp\left\{-U(x/\sigma)\right\}$$

  the partition function has the form

$$Z(\sigma) = \sigma^N Z(1)$$

- Using this result the ML estimate of σ is given by

$$\frac{\sigma}{N}\frac{d}{d\sigma}U(x/\sigma)\bigg|_{\sigma=\hat{\sigma}} - 1 = 0$$

- This equation can be solved numerically using any root finding method.

ML Estimation of σ for GGMRF's [108, 26]

- For a Generalized Gaussian MRF (GGMRF)

$$p(x) = \frac{1}{\sigma^N Z(1)} \exp\left\{ -\frac{1}{p\sigma^p} U(x) \right\}$$

where the energy function has the property that for all $\alpha > 0$

$$U(\alpha x) = \alpha^p U(x)$$

- Then the ML estimate of σ is

$$\hat{\sigma} = \left(\frac{1}{N} U(x) \right)^{(1/p)}$$

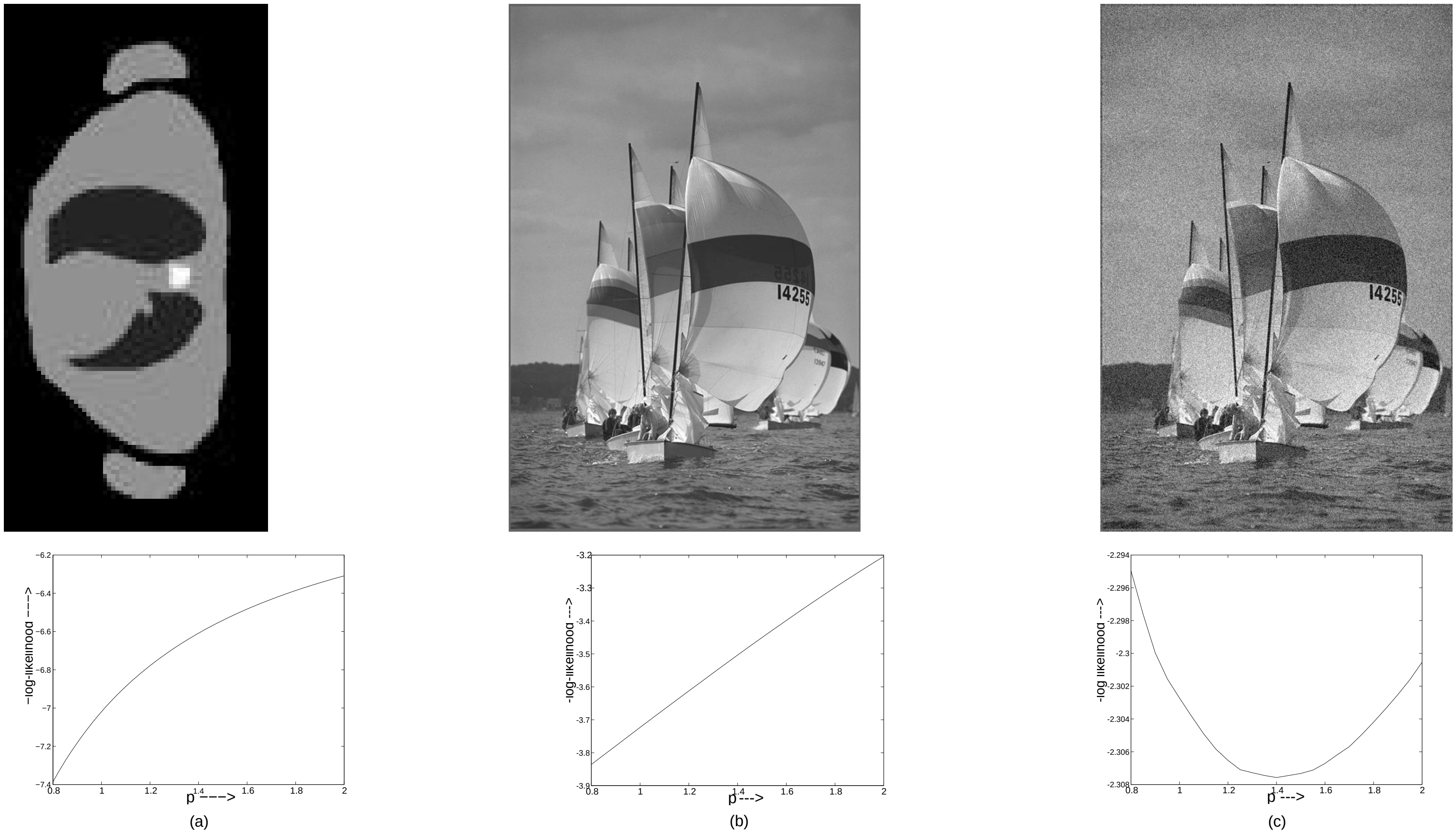
- Notice for that for the i.i.d. Gaussian case, this is

$$\hat{\sigma} = \sqrt{\frac{1}{N} \sum_s |x_s|^2}$$

Estimation of Temperature, T, and Shape, p, Parameters

- ML estimation of T[71]
  - Used to estimate T for any distribution.
  - Based on "off line" computation of log partition function.
- Adaptive method [133]
  - Used to estimate p parameter of GGMRF.
  - Based on measurement of kurtosis.
- ML estimation of p[145, 144]
  - Used to estimate p parameter of GGMRF.
  - Based on "off line" computation of log partition function.

Example Estimation of p Parameter



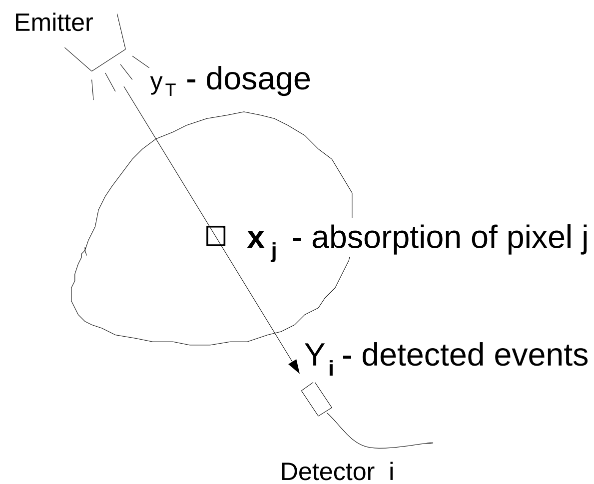
- ML estimation of p for (a) transmission phantom (b) natural image (c) image corrupted with Gaussian noise. The plot below each image shows the corresponding negative log-likelihood as a function of p. The ML estimate is the value of p that minimizes the plotted function.

Application to Tomography

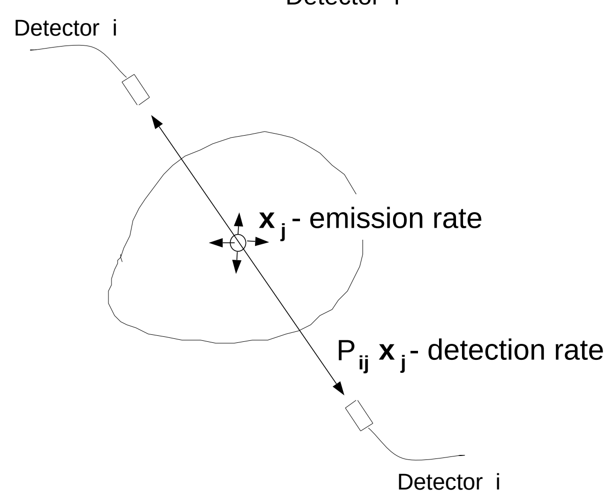
- Topics to be covered:
  - Tomographic system and data models
  - MAP Optimization
  - Parameter estimation

The Tomography Problem

- Recover image cross-section from integral projections
- Transmission problem



- Emission problem



Statistical Data Model[27]

- Notation
  - y - vector of photon counts
  - x - vector of image pixels
  - P - projection matrix
  - $P_{j,*}$ - j^{th} row of projection matrix
- Emission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-P_{i*}x \ + y_i \log\{P_{i*}x\} - \log(y_i!)\right)$$

- Transmission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-y_T e^{-P_{i*}x} + y_i(\log y_T - P_{i*}x) - \log(y_i!)\right)$$

- Common form

$$\boxed{\log p(y|x) = -\Sigma_{i=1}^{M} f_i(P_{i*}x)}$$

  - $f_i(\cdot)$ is a convex function
  - Not a hard problem!

Maximum A Posteriori Estimation (MAP)

- MAP estimate incorporates prior knowledge about image

$$
\begin{aligned}
\hat{x} &= \arg\max_{x} p(x|y) \\
&= \arg\max_{x>0} \left\{ -\sum_{i=1}^{M} f_i(P_{i*}x) - \sum_{k<j} b_{k,j}\,\rho(x_k - x_j) \right\}
\end{aligned}
$$

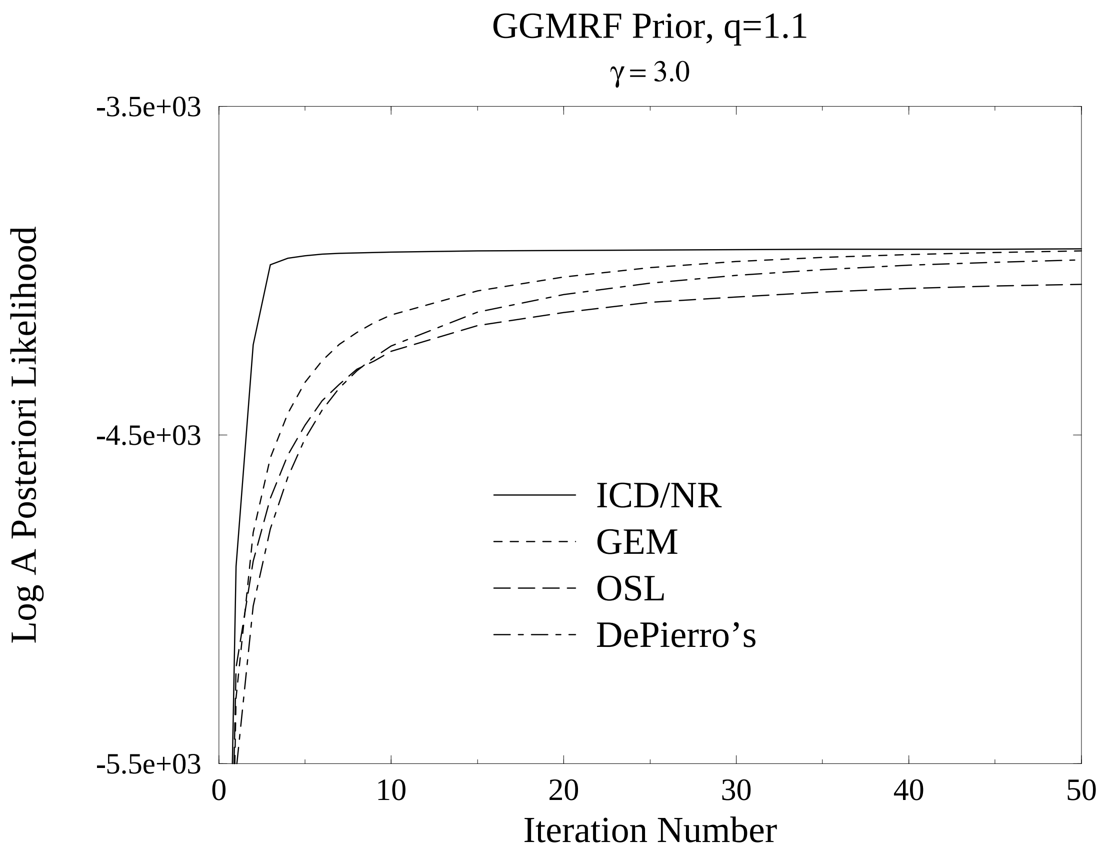
- Can be solved using direct optimization
- Incorporates positivity constraint

MAP Optimization Strategies

- Expectation maximization (EM) based optimization strategies
  - ML reconstruction[151, 107]
  - MAP reconstruction[81, 75, 84]
  - Slow convergence; Similar to gradient search.
  - Accelerated EM approach[59]
- Direct optimization
  - Preconditioned gradient descent with soft positivity constraint[45]
  - ICM iterations (also known as ICD and Gauss-Seidel)[27]

Convergence of ICM Iterations: MAP with Generalized Gaussian Prior $q = 1.1$

- ICM also known as iterative coordinate descent (ICD) and Gauss-Seidel



- Convergence of MAP estimates using ICD/Newton-Raphson updates, Green's (OSL), and Hebert/Leahy's GEM, and De Pierro's method, and a generalized Gaussian prior model with $q = 1.1$ and $\gamma = 3.0$.

Estimation of σ from Tomographic Data

- Assume a GGMRF prior distribution of the form

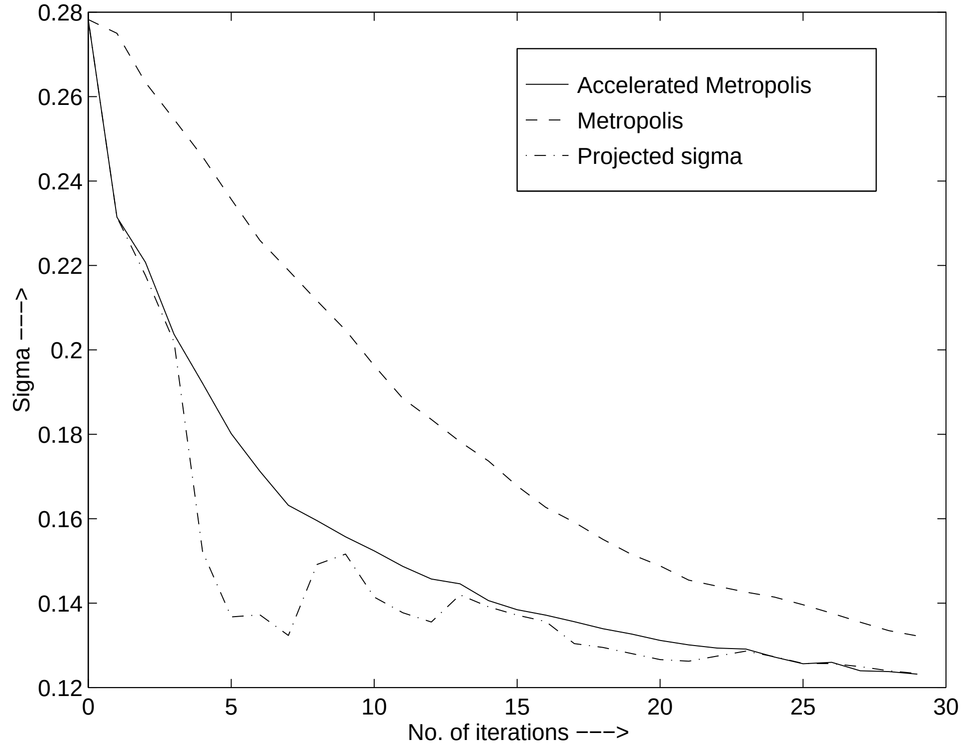
$$p(x) = \frac{1}{\sigma^N Z(1)} \exp\left\{ \frac{1}{p\sigma^p} U(x) \right\}$$

- Problem: We don't know X!
- EM formulation for incomplete data problem

$$\begin{aligned} \sigma^{(k+1)} &= \arg\max_{\sigma} E\left\{ \log p(X|\sigma) | Y = y, \sigma^{(k)} \right\} \\ &= \left(E\left\{ \frac{1}{N} U(X) | Y = y, \sigma^{(k)} \right\} \right)^{1/p} \end{aligned}$$

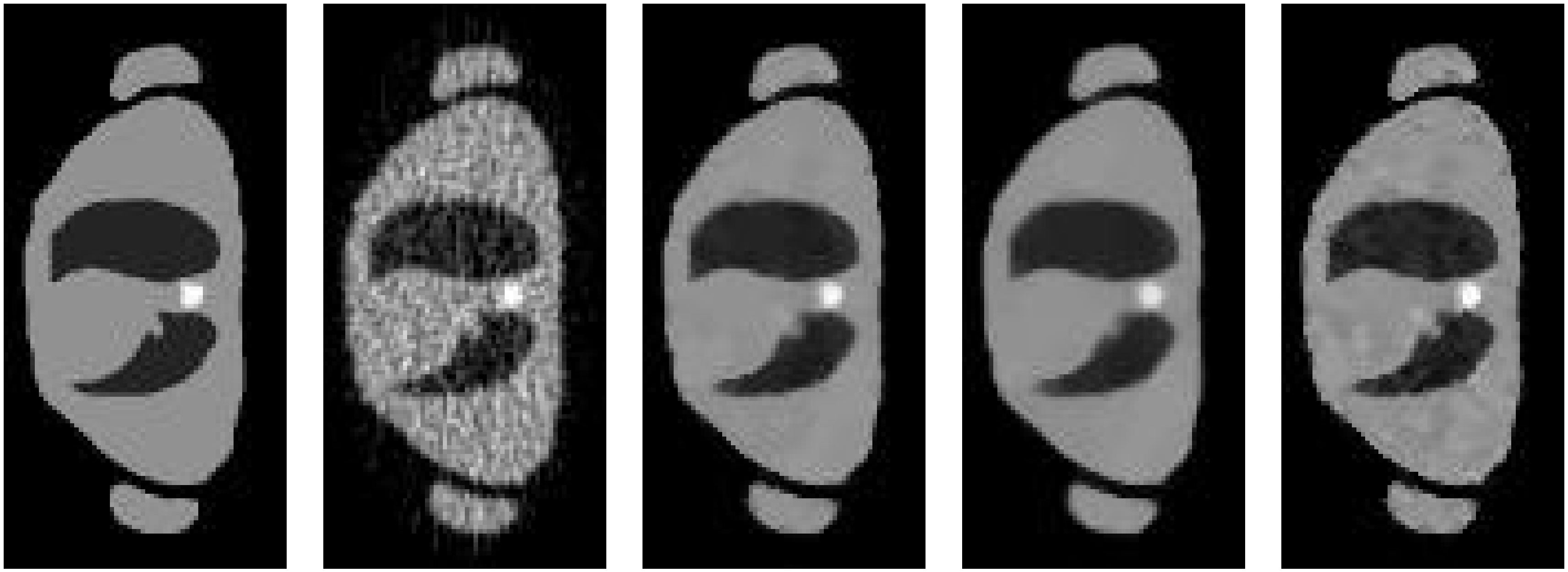
- Iterations converge toward the ML estimate.
- Expectations may be computed using stochastic simulation.

Example of Estimation of σ from Tomographic Data



- The above plot shows the EM updates for σ for the emission phantom modeled by a GGMRF prior ($p = 1.1$) using conventional Metropolis (CM) method, accelerated Metropolis (AM) and the extrapolation method. The parameter s denotes the standard deviation of the symmetric transition distribution for the CM method.

Example of Tomographic Reconstructions

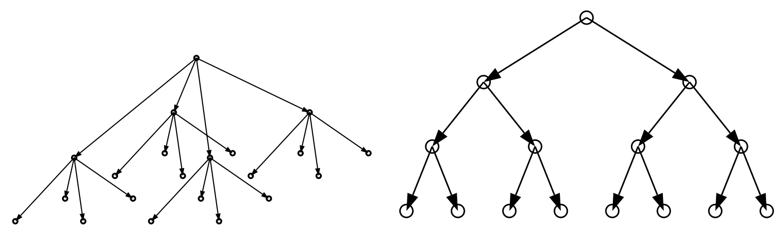


a	b	c	d	e

- (a) Original transmission phantom and (b) CBP reconstruction. Reconstructed transmission phantom using GGMRF prior with $p = 1.1$ The scale parameter σ is (c) $\hat{\sigma}_{ML} \approx \hat{\sigma}_{CBP}$, (d) $\frac{1}{2}\hat{\sigma}_{ML}$, and (e) $2\hat{\sigma}_{ML}$
- Phantom courtesy of J. Fessler, University of Michigan

Multiscale Stochastic Models

- Generate a Markov chain in scale



- Some references
  - Continuous models[12, 5, 111]
  - Discrete models[29, 111]
- Advantages:
  - Does not require a causal ordering of image pixels
  - Computational advantages of Markov chain versus MRF
  - Allows joint and marginal probabilities to be computed using forward/backward algorithm of HMM's.

Multiscale Stochastic Models for Continuous State Estimation

- Theory of 1-D systems can be extended to multiscale trees[6, 7].
- Can be used to efficiently estimate optical flow[111].
- These models can approximate MRF's[112].
- The structure of the model allows exact calculation of log likelihoods for texture segmentation[113].

Multiscale Stochastic Models for Segmentation[29]

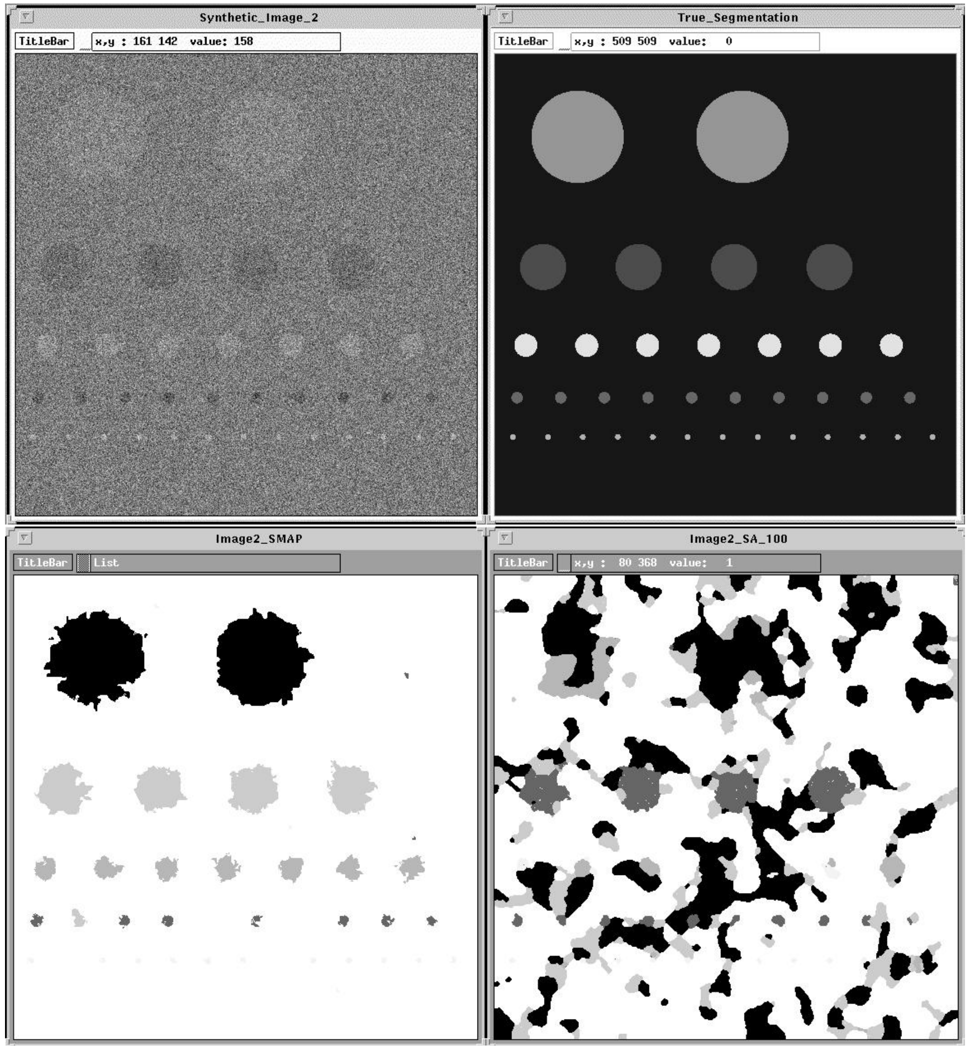
- Multiscale model results in non-iterative segmentation
- Sequential MAP (SMAP) criteria minimizes size of largest misclassification.
- Computational comparison

	Replacements per pixel				
	SMAP	SMAP + par. est.	SA 500	SA 100	ICM
image1	1.33	3.13	504	105	28
image2	1.33	3.55	506	108	28
image3	1.33	3.14	505	104	10

Segmentation of Synthetic Test Image

Synthetic Image

Correct Segmentation

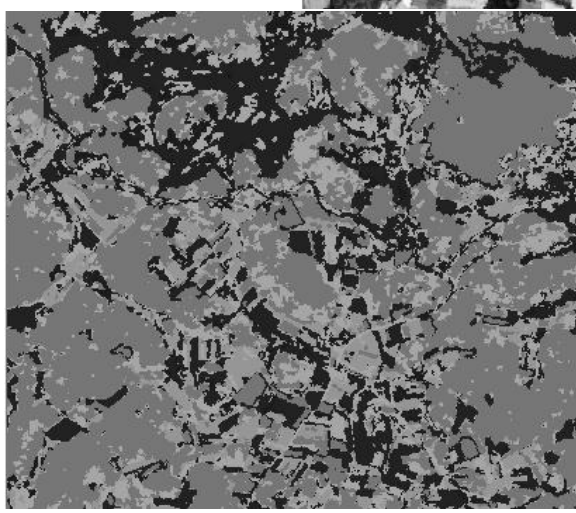


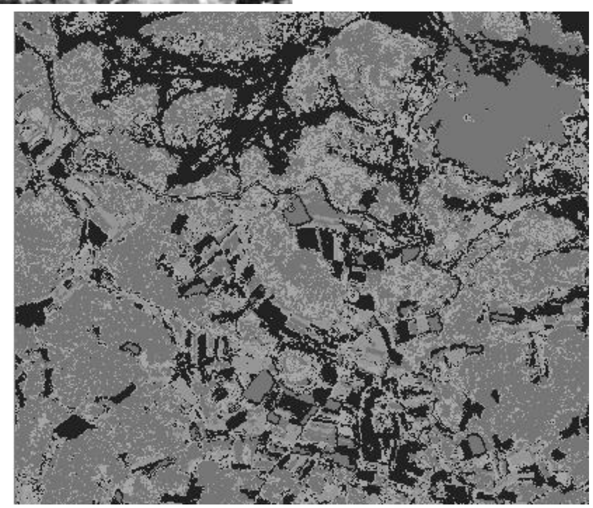
SMAP

100 Iterations of SA

Multispectral Spot Image Segmentation

SPOT image





SMAP Maximum Likelihood

High Level Image Models

- MRF's have been used to
  - model the relative location of objects in a scene[119].
  - model relational constraints for object matching problems[109].
- Multiscale stochastic models
  - have been used to model complex assemblies for automated inspection[166].
  - have been used to model 2-D patterns for application in image search[154].

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