# Primitive Recursion without Implicit Predecessor*

Michael J. Fischer† — *Yale University*
Robert P. Fischer‡ — *Hamden High School*
Richard Beigel§ — *Yale University*

**Abstract**

The *primitive recursive functions* are the least class of functions on the natural numbers containing the constant, successor, and projection functions, and closed under composition and the rule of primitive recursion. A weakened rule of primitive recursion, lacking implicit predecessor, is shown nevertheless to be adequate for defining all of the primitive recursive functions. However, it is conjectured that the still weaker rule of iteration is not adequate, and it is shown that an easily characterized class of 1-place functions that includes predecessor cannot be defined when only 0- and 1-place intermediate functions are allowed.

## 1 Rule of Primitive Recursion

The rule of *primitive recursion*[1], introduced by Gödel over 50 years ago [1, 2], defines a unique $n$-place function of the natural numbers $\phi$ in terms of an $(n-1)$-place function $\psi$ and an $(n+1)$-place function $\xi$ as follows:

$$\phi(y, x_2, \ldots, x_n) = \begin{cases} \psi(x_2, \ldots, x_n) & \text{if } y = 0; \\ \xi(y-1, \phi(y-1, x_2, \ldots, x_n), x_2, \ldots, x_n) & \text{if } y > 0. \end{cases}$$

*This work was supported in part by the National Science Foundation under grants CCR–8405478, CCR–8808949, CCR–8958528, and by the Office of Naval Research under contract N00014–89–J–1980.

†Address: Department of Computer Science, Box 2158 Yale Station, New Haven, CT 06520. E-mail: ⟨fischer-michael@cs.yale.edu⟩.

‡Address: 80 Killdeer Road, Hamden, CT 06517.

§Address: Department of Computer Science, Box 2158 Yale Station, New Haven, CT 06520. E-mail: ⟨beigel-richard@cs.yale.edu⟩.

1 Gödel uses the term "recursion" for what is now called "primitive recursion".

The class of *primitive recursive functions* can be defined to be the least class of functions containing the 0-place *zero function* $\mathsf{Z}() = 0$, the 1-place *successor function* $\mathsf{S}(x) = x + 1$, each $n$-place *projection function* $\pi_i^n(x_1, \ldots, x_n) = x_i$, $1 \leq i \leq n$, and closed under the rules of composition[2] and primitive recursion. (Cf. [3].)

In developing the primitive recursive functions, one typically shows that the 1-place *predecessor function* $\mathsf{P}(x) = \max(0, x - 1)$ is primitive recursive by observing that it follows immediately from starting functions $\mathsf{Z}$ and $\pi_1^2$ by the rule of primitive recursion. This definition uses the fact that a function defined by primitive recursion is given access to $y - 1$ when $y > 0$. Thus, predecessor is already implicit in the rule of primitive recursion, so one gets the predecessor function "for free". Predecessor is then used to define monus (natural number subtraction), which in turn is useful in defining many other arithmetic functions.

## 2 Weak Primitive Recursion

A natural question to ask is, "What happens if implicit predecessor is removed from the rule of primitive recursion?" We say that $\phi$ follows from $\psi$ and $\xi$ (of the same numbers of arguments as before) by the rule of *weak primitive recursion* if $\phi$ is the unique function satisfying

$$\phi(y, x_2, \ldots, x_n) = \begin{cases} \psi(x_2, \ldots, x_n) & \text{if } y = 0; \\ \xi(y, \phi(y-1, x_2, \ldots, x_n), x_2, \ldots, x_n) & \text{if } y > 0. \end{cases}$$

Under this rule, the value of $\phi(y, x_2, \ldots, x_n)$ when $y > 0$ is given by a previously-defined function $\xi$ that depends only on $\phi$'s arguments and on the value of $\phi$ at $y-1$. In particular, $\xi$ is not furnished with $y-1$. The class of *weak primitive recursive functions* is the least class of functions containing $\mathsf{Z}$, $\mathsf{S}$, and each $\pi_i^n$, and closed under composition and weak primitive recursion.

It is immediate from the definitions that if $\phi$ is defined from $\psi$ and $\xi$ by weak primitive recursion, then $\phi$ is defined from $\psi$ and $\xi'$ by (ordinary) primitive recursion, where $\xi'(y, z, x_2, \ldots, x_n) = \xi(\mathsf{S}(y), z, x_2, \ldots, x_n)$. $\xi'$ is primitive recursive if $\xi$ is; hence every weak primitive recursive function is primitive recursive.

[2] Let $n \geq 0$, $k \geq 1$. An $n$-place function $\phi$ is the composition of a $k$-place function $\psi$ with $n$-place functions $\xi_1, \ldots, \xi_k$, written $\psi \circ (\xi_1, \ldots, \xi_k)$, if $\phi(x_1, \ldots, x_n) = \psi(\xi_1(x_1, \ldots, x_n), \ldots, \xi_k(x_1, \ldots, x_n))$.

Conversely, if $\phi$ is defined from $\psi$ and $\xi'$ by ordinary primitive recursion, then $\phi$ is defined from $\psi$ and $\xi''$ by weak primitive recursion, where $\xi''(y, z, x_2, \ldots, x_n) = \xi'(\mathsf{P}(y), z, x_2, \ldots, x_n)$. If we could show that $\mathsf{P}$ is weak primitive recursive, then it would follow that $\xi''$ is weak primitive recursive if $\xi'$ is, and hence the class of weak primitive recursive functions would coincide with the class of ordinary primitive recursive functions.

## 3 Weak Primitive Recursive Functions

We show that the class of weak primitive recursive functions coincides with the class of primitive recursive functions. To begin our development, we note that $\mathsf{K}^n_c(x_1, \ldots, x_n)$, the $n$-place constant function with value $c$, is weak primitive recursive for all $n, c \geq 0$. The usual primitive recursive definition of $\mathsf{K}^n_c$ from $\mathsf{Z}$ and $\mathsf{S}$, while employing the rule of primitive recursion, does not use the stage number and hence works equally well with the rule of weak primitive recursion. (Cf. [3].)

To show that the weak primitive recursive functions are closed under the rule of primitive recursion (and hence are the same as the primitive recursive functions), we show that $\mathsf{P}$ is weak primitive recursive. We proceed by exhibiting weak primitive recursive definitions of a number of auxiliary functions. For each, we describe the function and show how it is obtained from starting functions and functions previously shown to be weak primitive recursive.

1. $+$ (ordinary 2-place addition):

$$x + y = \begin{cases} y & \text{if } x = 0; \\ \mathsf{S}((x-1) + y) & \text{if } x > 0. \end{cases}$$

   Note that this definition fits the rule of weak primitive recursion, for the recursive case depends only on the previous value of the function (i.e. the value of $+$ at arguments $(x-1)$ and $y$) and not directly on the previous stage number $(x-1)$.

2. Let

$$\mathsf{sg}(x) = \begin{cases} 0 & \text{if } x = 0; \\ 1 & \text{if } x > 0. \end{cases}$$

   $\mathsf{sg}$ is obtained from constant functions by the rule of weak primitive recursion.

3. Let

$$\overline{\mathsf{sg}}(x) = \begin{cases} 1 & \text{if } x = 0; \\ 0 & \text{if } x > 0. \end{cases}$$

$\overline{\mathsf{sg}}$ is obtained from constant functions by the rule of weak primitive recursion.

4. Let the $\mathsf{even}$ predicate be defined by

$$\mathsf{even}(x) = \begin{cases} 1 & \text{if } x \text{ is even;} \\ 0 & \text{if } x \text{ is odd.} \end{cases}$$

$\mathsf{even}$ can be defined recursively by the scheme

$$\mathsf{even}(x) = \begin{cases} \mathsf{K}^0_1() & \text{if } x = 0; \\ \overline{\mathsf{sg}}(\mathsf{even}(x-1)) & \text{if } x > 0. \end{cases}$$

Since the recursive case depends only on the previous value of the function, $\mathsf{even}$ is primitive recursive.

5. Let $\mathsf{div2}(x) = \lfloor x \div 2 \rfloor$. Note that for $x > 0$,

$$\mathsf{div2}(x) = \begin{cases} \mathsf{div2}(x-1) & \text{if } x \text{ is odd;} \\ \mathsf{div2}(x-1) + 1 & \text{if } x \text{ is even.} \end{cases}$$

Hence, a weak primitive recursive definition is

$$\mathsf{div2}(x) = \begin{cases} \mathsf{Z}() & \text{if } x = 0; \\ \mathsf{div2}(x-1) \mathbin{+} \mathsf{even}(x) & \text{if } x > 0. \end{cases}$$

6. Let "greater than 1" be the predicate defined by

$$>\!\mathsf{1}(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } x = 1; \\ 1 & \text{if } x > 1. \end{cases}$$

Then $>\!\mathsf{1} = \mathsf{sg} \circ \mathsf{div2}$ is the composition of two weak primitive recursive functions.

7. $\mathsf{P}(x) = \max(0, x-1)$. A weak primitive recursive definition is

$$\mathsf{P}(x) = \begin{cases} \mathsf{Z}() & \text{if } x = 0; \\ \mathsf{P}(x-1) \mathbin{+} >\!\mathsf{1}(x) & \text{if } x > 0. \end{cases}$$

Note that the recursive definitions of both $\mathsf{div2}$ and $\mathsf{P}$ *do* refer to the stage number, but both use the number of the current stage $x$ rather than the number of the previous stage $x-1$ as provided by ordinary primitive recursion.

The above establishes:

**Theorem 1** *The class of weak primitive recursive functions is the same as the class of primitive recursive functions.*

## 4 Iterative Functions

At this point, one might ask, "Is the stage number really necessary at all in order to define the primitive recursive functions?" We say that an $n$-place function $\phi$ follows from an $(n-1)$-place function $\psi$ and an $n$-place function $\xi$ by the rule of *iteration* if $\phi$ is the unique function satisfying

$$\phi(y, x_2, \ldots, x_n) = \begin{cases} \psi(x_2, \ldots, x_n) & \text{if } y = 0; \\ \xi(\phi(y-1, x_2, \ldots, x_n), x_2, \ldots, x_n) & \text{if } y > 0. \end{cases}$$

The class of *iterative functions* is defined like the class of primitive recursive functions except the rule of iteration replaces the rule of primitive recursion.

We conjecture that the iterative functions are a proper subclass of the primitive recursive functions. In particular, we conjecture that for every 1-place iterative function $f(x)$, there exists a number $b$ such that for all $x$, $f(x)$ does not lie in the open interval $(b, x)$. This would imply that $\mathsf{div2}$ and $\mathsf{P}$ and a large number of other interesting functions were not iterative.

The conjecture *does* hold if one removes all of the functions $\pi_i^n$ for $n \geq 2$ from the set of starting functions. In that case, only 0- and 1-place functions can be defined, and we call the resulting class the *simple iterative functions*.

**Theorem 2** *For every 1-place simple iterative function $f(x)$, there exists a constant $b \geq 0$ such that for all $x$, either $f(x) \leq b$ or $f(x) \geq x$.*

*Proof:* The starting functions $\mathsf{Z}$, $\mathsf{S}$, and $\pi_1^1$ all satisfy the theorem for $b = 0$.

Suppose $f$ is a 1-place function defined by composition from previously-defined simple iterative functions $g$ and $h$. Then $g$ and $h$ are both 1-place, so $f(x) = g(h(x))$. By induction, the theorem holds for $g$ and $h$ with constants $b_g$ and $b_h$, respectively. Let $b_f = \max(\{b_g\} \cup \{g(y) \mid y \leq b_h\})$. We show the theorem holds for $f$ with constant $b_f$.

Choose $x$ and let $y = h(x)$.

*Case 1:* $y \le b_h$. Then $f(x) = g(y) \le b_f$ by definition of $b_f$.
*Case 2:* $y > b_h$. Then $y \ge x$ since the theorem holds for $h$ with constant $b_h$. Since the theorem also holds for $g$ with constant $b_g$, then either $g(y) \le b_g$ or $g(y) \ge y$. Hence, either $f(x) \le b_g \le b_f$ or $f(x) \ge y \ge x$.

In both cases, the theorem holds for $f(x)$.

Now, suppose $f$ is a 1-place function defined by iteration from previously-defined simple iterative functions $g$ and $h$. Then $g$ is a 0-place function and $h$ is a 1-place function, so by induction, the theorem holds for $h$ with constant $b_h$. Since $f$ satisfies:

$$f(x) = \begin{cases} g() & \text{if } x = 0; \\ h(f(x-1)) & \text{if } x > 0; \end{cases}$$

it follows that $f(0) = h^x(g())$, where $h^x$ denotes the $x$-fold composition of $h$ with itself.

We consider two cases.

*Case 1:* $f$ is 1-1. Then in every closed interval $[0, a]$, there is some $x_a$ such that $f(x_a) \ge a \ge x_a$. It follows that the set $\{x_a \mid a \in \mathsf{N}\}$ is infinite; hence there are infinitely many $x$ for which $f(x) \ge x$. Since $f$ is 1-1, there are only finitely many $x$ for which $f(x) \le b_h$. Hence, we may choose $\overline{x}$ satisfying $f(\overline{x}) \ge \overline{x}$ and larger than all $x$ for which $f(x) \le b_h$.

We show by induction on $x$ that $f(x) \ge x$ for all $x \ge \overline{x}$. It's true by definition for $x = \overline{x}$. Assume for the inductive step that $x > \overline{x}$ and that $y = f(x-1) \ge x - 1$. Since $h(y) = f(x) > b_h$ and the theorem holds for $h$, then $h(y) \ge y$. But if $h(y) = y$, then $f(x) = f(x-1)$, contradicting the assumption that $f$ is 1-1. Hence, $f(x) = h(y) > y \ge x - 1$, so $f(x) \ge x$ as desired. It follows by induction that $f(x) \ge x$ for all $x \ge \overline{x}$.

Let $b_f = \max\{f(x) \mid x < \overline{x}\}$. Then $f(x) \le b_f$ for all $x < \overline{x}$. Hence, the theorem holds for $f$.

*Case 2:* $f$ is not 1-1. Let $x < x'$ be such that $f(x) = f(x')$. Then

$$f(x+1) = h(f(x)) = h(f(x')) = f(x'+1),$$

and by a straightforward induction, $f(x+k) = f(x'+k)$ for all $k \ge 0$. But this implies that $f$ is ultimately periodic with period at most $x' - x$; hence $f$ is bounded. Therefore, we can choose $b_f = \max_x f(x)$ and the theorem holds for $f$. ∎

We do not know how to generalize the conjecture to the full class of iterative functions, and it is conceivable, for example, that $\mathsf{P}(x) = g(x, x)$ for some 2-place iterative function.

Lest one believe that the class of iterative functions is trivial, note that addition ($+$), multiplication ($\times$), $\mathsf{sg}$, $\overline{\mathsf{sg}}$, and $\mathsf{even}$ are easily shown to be iterative as well as some less obvious functions such as $\mathsf{mod}_k(x) = (x \bmod k)$ and $>\!\mathbf{1}(x)$.

The functions $\mathsf{mod}_1, \mathsf{mod}_2, \mathsf{mod}_3, \ldots$ can be defined inductively as follows: $\mathsf{mod}_1 = \mathsf{K}_0^1$, the constant function 0. Now assume we have already defined $\mathsf{mod}_{k-1}$. Let

$$\mathsf{cycle}_k(x) = \mathsf{mod}_{k-1}(x) + sg(\mathsf{mod}_{k-1}(x)) + \overline{\mathsf{sg}}(x).$$

$\mathsf{cycle}_k$ has the property that $\mathsf{cycle}_k(x) = (x+1) \bmod k$ for $x \leq k-1$. (The values of $\mathsf{cycle}_k(x)$ for $x \geq k$ do not matter for our purposes.) Now, $\mathsf{mod}_k$ is defined by the following iteration:

$$\mathsf{mod}_k(x) = \begin{cases} 0 & \text{if } x = 0 \\ \mathsf{cycle}_k(\mathsf{mod}_k(x-1)) & \text{if } x > 0. \end{cases}$$

To define $>\!\mathbf{1}(x)$, we make a series of definitions:

- $p(x) = (x+2) \times \mathsf{even}(x)$. Successive values of $p$ are 2, 0, 4, 0, 6, 0, 8, 0, ....
- $q(x) = p^x(1)$ is defined by iteration. Successive values of $q$ are 1, 0, 2, 4, 6, 8, 10, 12, ....
- $>\!\mathbf{1}(x) = \mathsf{sg}(q(x)) \times \mathsf{sg}(x)$.

## References


[1] Kurt Gödel. Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik*, 38:173–198, 1931.

[2] Kurt Gödel. On formally undecidable propositions of Principia Mathematica and related systems I. In Martin Davis, editor, *The Undecidable*, pages 4–38. Raven Press, Hewlett, New York, 1965. English translation of [1].

[3] Harry R. Lewis and Christos H. Papadimitriou. *Elements of the Theory of Computation*. Prentice-Hall, Inc., 1981.