Stable Matching Beyond Bipartite Graphs

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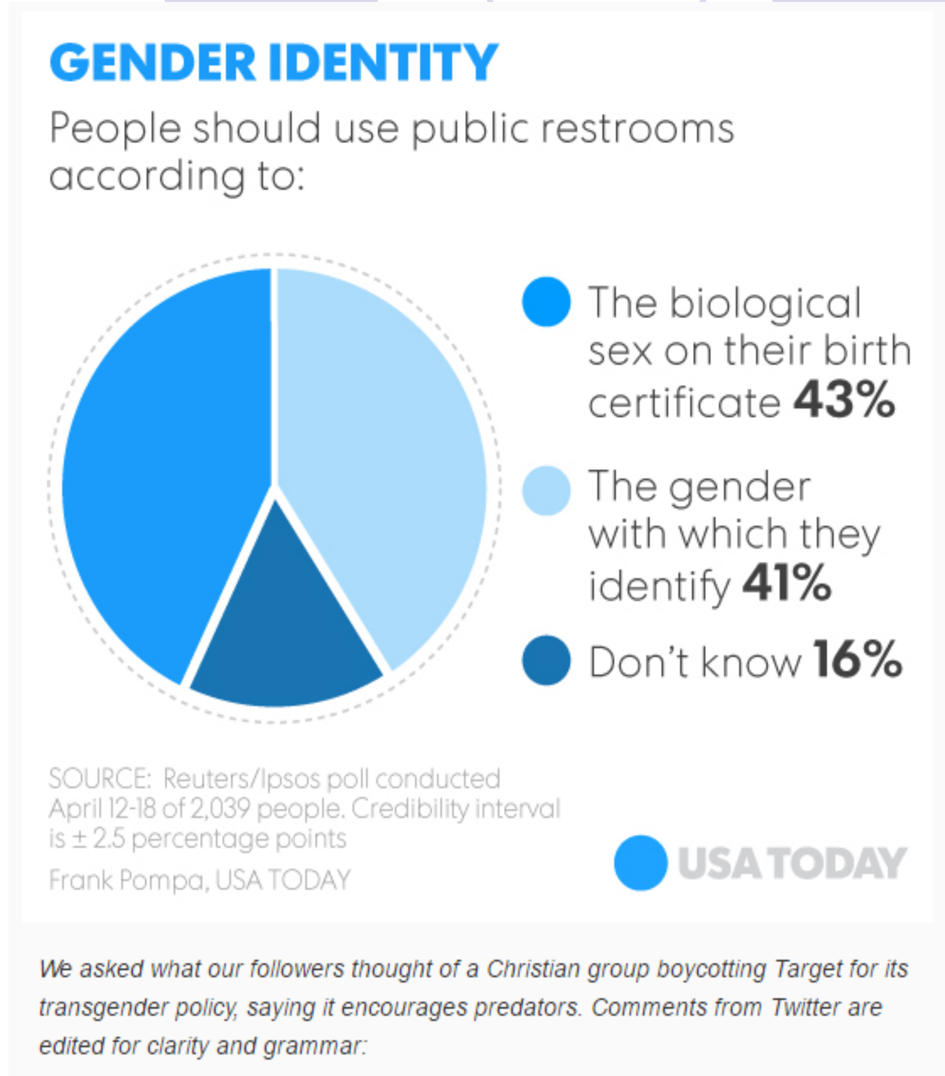
Road Map

- Introduction
- Stable Marriage Problem (SMP)
- Gale-Shapley (GS) algorithm
- Stable binary matching with multiple genders
- Stable K-ary matching with multiple genders
- Extensions
- Conclusion

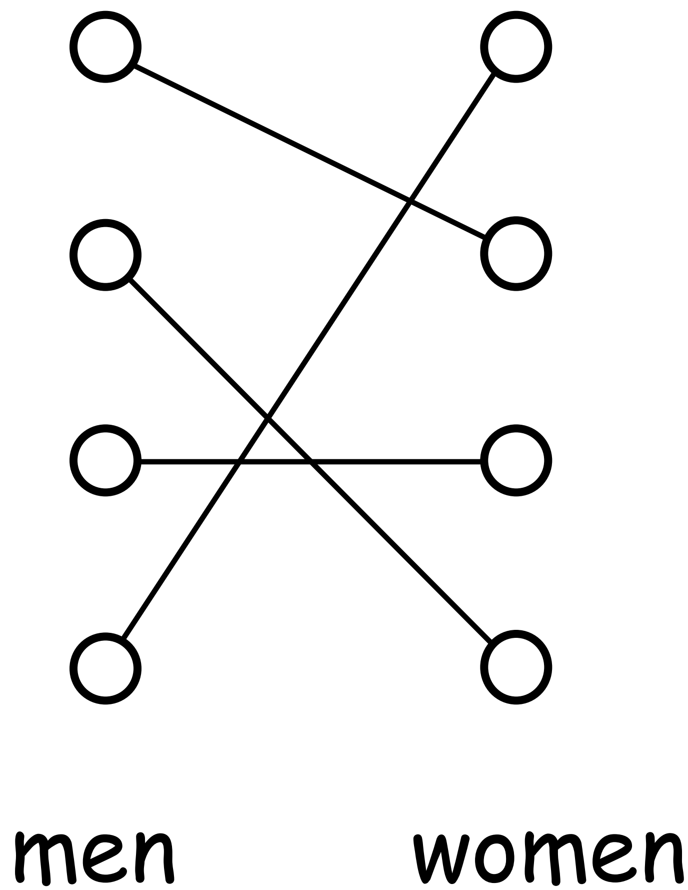
Introduction

- Restroom Rules
  - North Carolina
  - Donald Trump
  - Target's policy
- Sex vs. gender
- Stable matching with multiple genders
  - Binary matching
  - K-ary matching



Stable Marriage Problem (SMP)

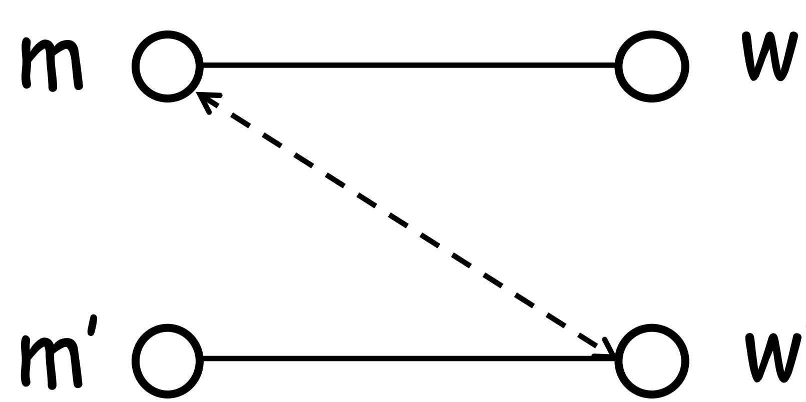
- Perfect matching



- Stable matching:

does not exist

  - m of the first pair prefers w' over w, and
  - w' of the second pair prefers m over m'



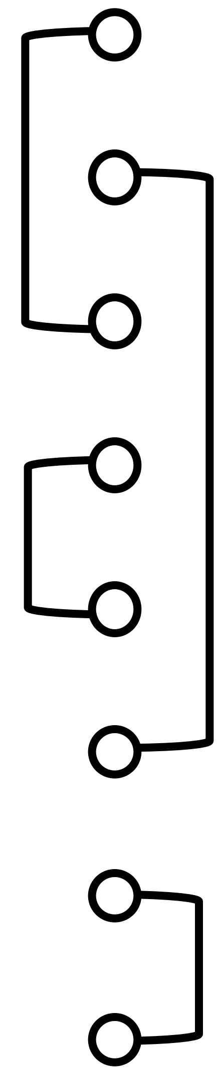
Gale-Shapley (GS) Algorithm

- Each person has his/her preference list.
- GS algorithm
  1. Each unengaged man proposes to the woman he prefers most.
  2. Each woman replies “maybe” to her suiter she most prefers, and replies “no” to all others. (She is then provisionally “engaged”.)
  3. If all men are engaged, stop; otherwise, each unengaged man proposes to the most-preferred woman he has not yet proposed.
  4. Go to step 2.

1. Complexity: n^2 (n: number of elements in a gender)

Some Well-known Extensions

1. Stable roommate problem
   1. Single gender
2. College admission problem
   1. Multiple matchings
3. Hospital/residents problem
   1. With couples

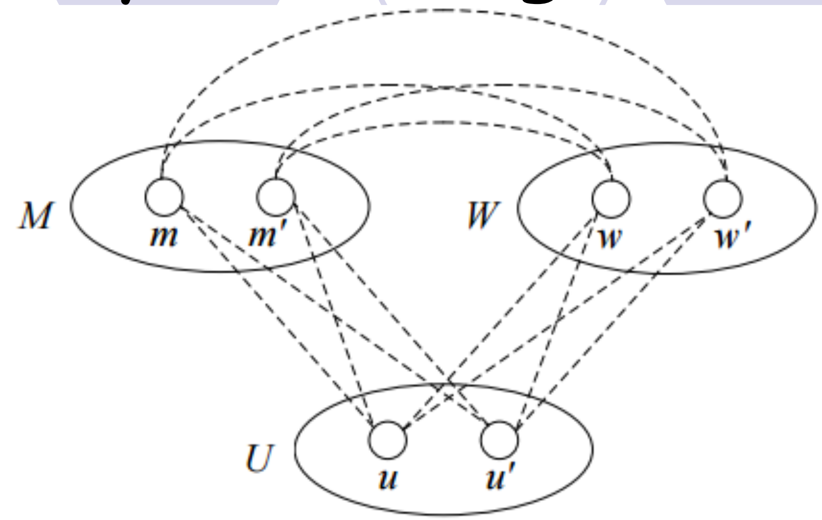


Multiple Genders: k-nary matching

M: men

W: women

U: undecided



Blocking family: if each member prefers each member of *that* family to its *current* family

Example 1: current matching is {(m, w, u), (m', w', u')}

(m', w, u) is a blocking family if m' prefers w and u and both w and u prefers m'

Stable Binary Matching with Multi-Genders

Theorem 1: There exists preference lists under which there exists no stable binary matching with k (≠2) genders.

Note

- Result holds even if self-matching is allowed, as in U.
- Stable roommate solution can be used to find one if it exists.

$$\{(m,w),(m',u),(w',u')\} \quad \{(m,w),(m',u'),(w',u)\}$$
$$\{(m,w'),(m',u),(w,u')\} \quad \{(m,w'),(m',u'),(w,u)\}$$
$$\{(m,u),(m',w),(w',u')\} \quad \{(m,u),(m',w'),(w,u')\}$$
$$\{(m,u'),(m',w),(w',u)\} \quad \{(m,u'),(m',u),(u',w)\}$$

Stable k-ary Matching with Multi-Genders

K-ary matching: $(u_1, u_2, \ldots, u_k)$ (k: the number of genders)

Iterative Binding: Iteratively apply GS to pair wisely and bind all disjoint sets through a spanning tree.

Algorithm 1 Iterative Binding GS Algorithm

/* I is a gender set with $|I| = k$ */;
T (binding tree) and P (matching pairs) are empty;
while T is not a spanning tree on I **do**
    Find i, $j \in I$: (i, j) does not cause a cycle in T;
    $V(T) = V(T) \cup \{i, j\}$; $E(T) = E(T) \cup \{(i, j)\}$;
    $P = P \cup \text{GS}(i, j)$;
Derive E, equivalence classes from equivalence relation $(-, -)$ "in the same matching tuple" on P;
return E (matching k-tuples)

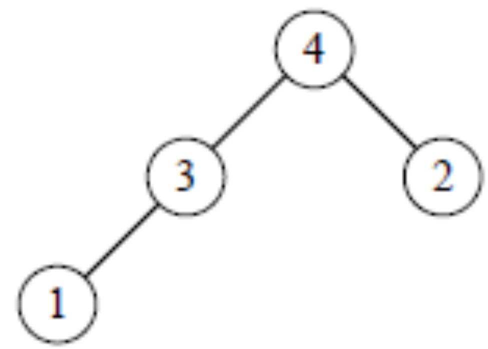
Stable k-ary Matching

Theorem 2: The iterative GS constructs a stable k-ary matching.

Theorem 3: The k-1 rounds of the binding process is tight.

Note

- k^{k-1} binding trees
- $(k-1)n^2$ iterations of pairwise matching

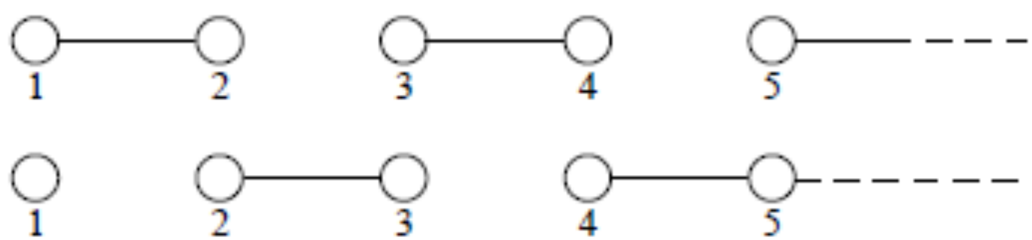


Extension: Parallel Implementation

Theorem 4: Using EREW PRAM, the iterative GS takes at most Δn^2 iteration, where Δ is the maximum node degree.

Note

- When Δ=2, two rounds are needed (even-odd matching)



- Under CREW PRAM, binding can be done simultaneously.
- (CREW PRAM can be emulated under EREW PRAM through log Δ rounds of data replication.)

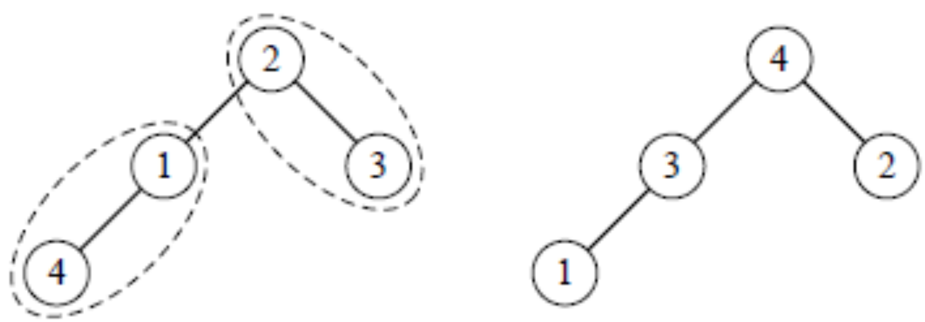
Extension of Unstable Condition

In a blocking family, the lead member of components from the same family decides the subgroup preference.

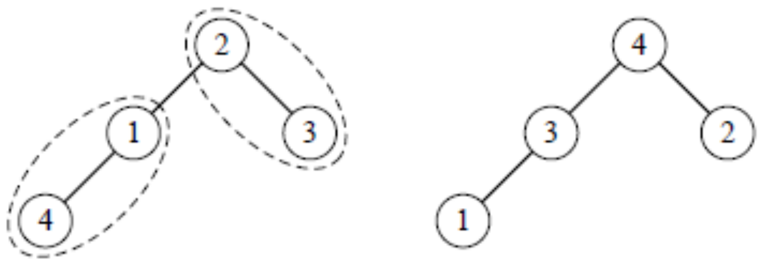
(In Example 1, w and u form a subgroup. If W has a higher priority than U, w decides for u.)

Bitonic sequence: it monotonically increases and then monotonically decreases, e.g., (1, 3, 4, 2) and (4, 3, 2, 1).

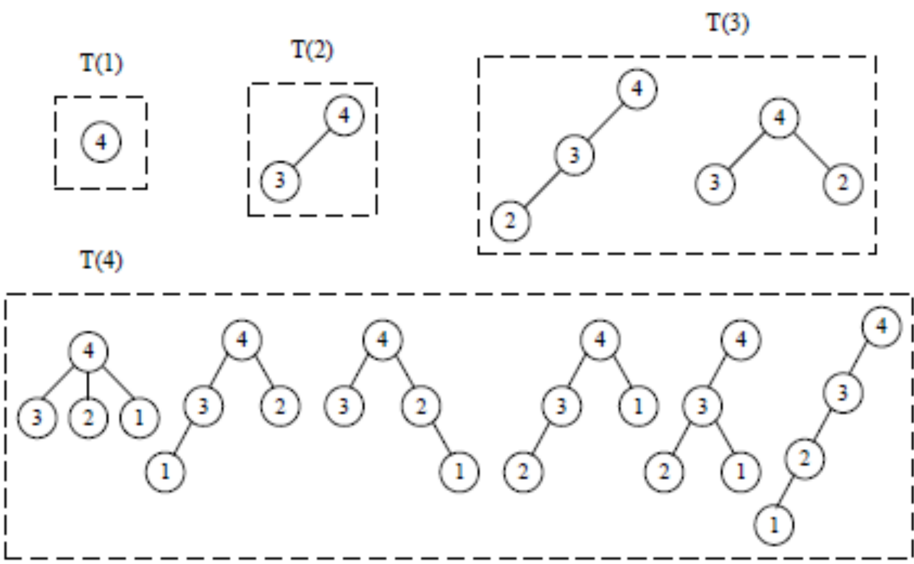
Bitonic tree: if any two nodes in the tree is connected through a path that is a bionic sequence.



Priority-Based Iterative GS



Number of priority tree: (k-1)!



Algorithm 2 Priority-Based Iterative Binding GS Algorithm

/* I is a gender set with $|I| = k$ and i_{max} is the highest priority gender */;
$V(T) = \{i_{max}\}$, $E(T) = P = \{\}$, and $I = I - \{i_{max}\}$;
while I is not empty **do**
  Select an i in $V(T)$ and j in I with the highest priority;
  $V(T) = V(T) \cup \{j\}$; $E(T) = E(T) \cup \{(i, j)\}$;
  $I = I - \{j\}$;
  $P = P \cup \text{GS}(i, j)$;
Derive E, equivalence classes from equivalence relation $(-,-)$ "in the same matching tuple" on P;
return E (matching k-tuples);

Conclusions

- Stable matching with k genders
  - binary matching (negative result)
  - k-ary matching (positive result): iterative GS
- Two extensions
  - Parallel implementation of iterative GS
  - Extension of unstable condition
- Future work
  - Other possible weakened blocking family

Special Thanks

- The WeChat group of the classmates from Shanghai Guling No. 1 Elementary School
- Inspired by the discussion on matching making in the group discussion