Asymptotic Analysis on an Unbounded Zero-One Knapsack with Discrete-Sized Objects*

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Abstract

The problem being discussed in this paper is a special case of the unbounded knapsack problem :

$$\begin{array}{ll} \max & z_n(M) = \frac{1}{n}\sum_{i=1}^{n} p_i x_i \\ \text{s.t.} & \sum_{i=1}^{n} c_i x_i \leq \beta_0 n \\ & x_i \in \{0,1\}\ \forall i = 1,\ldots,n; \end{array}$$

where p_i's are uniformly distributed random variables in $[0,1]$, and c_is are discrete random variables distributed uniformly in $\{1/M, 2/M, \ldots, (M-1)/M, 1\}$. Assuming that M is large, it is shown that $\lim_{n\to\infty} z_n(M)$ approximately equals to $\sqrt{\frac{2\beta_0}{3}} - 0.3062(\sqrt{\beta_0}M)^{-1}$.

1 Introduction

The unbounded knapsack problem can be stated as the following combinatorial optimization problem :

$$\begin{array}{ll} \max & z_n = \frac{1}{n}\sum_{i=1}^{n} p_i x_i \\ \text{s.t.} & \frac{1}{n}\sum_{i=1}^{n} c_i x_i \leq \beta_0 \\ & x_i \in \{0,1\}\ \forall i = 1,\ldots,n. \end{array}$$

From Lai [1], Marchetti-Spaccamela and Vercellis [4] and Lueker [3], the optimal solution of the above problem is equivalent to the solution of the following problem for large n.

$$\begin{array}{ll} \max & \lim_{n\to\infty} z_n = \int_0^{1/m^*}\int_{m^*c}^{1} p\ dp\ dc \\ \text{s.t.} & \int_0^{1/m^*}\int_{m^*c}^{1} dp\ dc = \beta_0 \\ & \text{where } 1 \leq m^* < \infty. \end{array} \quad (1)$$

By making assumption on the statistical distributions for p_is and c_is, the average profit gain can thus be obtained.

It should be noted the assumption that c_i's are continuous can hardly be found in practice, particularly in multi-unit combinatorial auction [2, 5]. An auctioneer has a number of products, say notebook computers, for auction off. Bidders submit their offers. Once the auction time has closed, the auctioneer selectively allocates the products to the bidders in order to optimize the profit. Since the bid size is discrete, the average profit gain can only be evaluated through an approximation.

In this paper, we extend the work in [4] and [3]. Specifically, the average profit gain obtained by the following problem, (2), will be analyzed.

$$\begin{array}{ll} \max & z_n(M) = \frac{1}{n}\sum_{i=1}^{n} p_i x_i \\ \text{s.t.} & \sum_{i=1}^{n} c_i x_i \leq \beta_0 n \\ & x_i \in \{0,1\}\ \forall i = 1,\ldots,n; \end{array} \quad (2)$$

where p_i's are uniformly distributed random variables in $[0,1]$, and c_is are discrete random variables distributed uniformly in $\{1/M, 2/M, \ldots, (M-1)/M, 1\}$. First, the average profit gain for the case that c_i's are continuous will be derived. Assuming that each bidder whose original bid size is in $[j/M, (j+1)/M]$ will bid for either j/M or $(j+1)/M$ equally random, the average profit gain can be derived.

The rest of the paper will elucidate the derivation and an application in multi-unit combinatorial auction. The average profit gain for the case that c_i's are continuous random variables will be derived in the next section. The average profit gain for discrete c_i's will be derived in Section 3. Section 4 presents an application of the result in multi-unit combinatorial auction. Finally, the conclusion will be presented in Section 5.

2 Average profit gain : Continuous c_i's

For the case that c_i's are continuous random variables distributed uniformly in $[0,1]$, the optimal solution $\lim_{n\to\infty} \hat{z}_n$ can be obtained by simply solving the second equality constraint in (1) for m^* and, then, substituting the value to the first equality for the optimal $\lim_{n\to\infty} \hat{z}_n$. Solving the second equality in (1), we are

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able to show that

$$m^* = \sqrt{\frac{1}{6\beta_0}}. \quad (3)$$

Putting this result back to the first equality in (1), we have

$$\lim_{n\to\infty} \hat{z}_n = \sqrt{\frac{2\beta_0}{3}}. \quad (4)$$

This result provides the researchers the first approximation on the average profit being gained from each object. But the assumption is that the product is divisible, i.e., the size of each object can be in any fractional number.

3 Average profit gain : Discrete c_i's

Let us consider a simple multi-unit combinatorial auction problem. Suppose an auctioneer has 1 000 000 units of notebook computers for auction off. The auctioneer anticipates that 2 000 bidders will come. Moreover, the auctioneer also expects that their bid sizes are uniformly random in $[1, 1\ 000]$ and the bid price is another independent random variables uniformly in $[100, 100\ 000]$. Accordingly, the average profit gain is estimated as follows :

$$\begin{aligned} 2\ 000 \times 100\ 000 \times \hat{z}_{2000} &\approx 2\ 000 \times 100\ 000 \times \sqrt{\frac{2(0.5)}{3}} \\ &= 115.47 \times 10^6. \end{aligned}$$

Since the quantized levels in both bid size and bid price are small, 0.001 and 0.001 respectively, the approximation is close to the actual value. However, it will be inaccurate if the quantized level is not small.

Let M be the maximum size that a bidder will bid. The actual inequality constraint will be expressed as follows :

$$\frac{1}{n}\sum_{i=1}^{n} c_i M x_i \leq \beta_0 M. \quad (5)$$

Here $c_i M \in \{1, 2, \ldots, M\}$, is the actual bid size for the i^{th} bidder. Let $\lim_{n\to\infty} \hat{z}_n(M)$ be the optimal average profit gain,

$$\lim_{n\to\infty} \hat{z}_n(M) \leq \lim_{n\to\infty} \hat{z}_n.$$

Equality holds when $M \to \infty$. Besides, let us define the percentage error, $E(M)$, as follows :

$$E(M) = \frac{\lim_{n\to\infty} \hat{z}_n - \lim_{n\to\infty} \hat{z}_n(M)}{\lim_{n\to\infty} \hat{z}_n}. \quad (6)$$

We now consider the case that the probability distribution for the bid size c, $P(c)$, is uniformly distributed in the set $\{1/M, 2/M, \ldots, 1\}$, that is,

$$\begin{aligned} P(c) &= \sum_{k=1}^{M} \frac{1}{M}\delta(c, k/M) \quad (7) \\ \delta(c, k/M) &= \begin{cases} 1 & \text{if } c = k/M \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

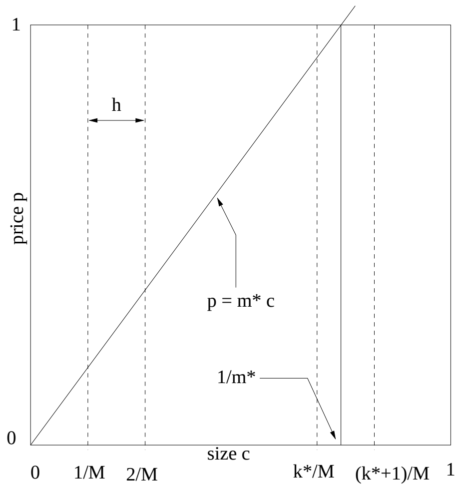

Figure 1: The straight line $p = m^* c$ is the decision boundary for product allocation. All the bids (p, c) above this line will be allocated.

In practice, we assume that each bidder whose original bid size is in the range $[j/M, (j+1)/M]$ ($j = 0, 1, \ldots, M-1$) will bid up to $(j+1)/M$. Therefore, the equality constraint in Equation (1) can be written as follows :

$$\sum_{i=1}^{k^*} \left(1 - i\frac{m^*}{M}\right)\left(\frac{i}{M^2}\right) = \beta_0 \quad (8)$$

where m^* is the slope of the straight line for the decision boundary, see Figure 1, and m^* and k^* can be related by the following inequatlity.

$$\frac{k^*}{M} < \frac{1}{m^*} < \frac{k^*+1}{M}.$$

Solving Equation (8), it is able to obtain

$$\frac{k^*(k^*+1)}{2} - \frac{m^*}{M}\frac{k^*(k^*+1)(2k^*+1)}{6} = \beta_0 M^2. \quad (9)$$

Since the number of bidders n and the number of quantized intervals M are large, it can be assumed that $k^* \gg 1$ and $m^*/M \approx k^*$. Hence,

$$k^* \approx \sqrt{6\beta_0}\, M. \quad (10)$$

The average profit gain $\lim_{n\to\infty} \hat{z}_n(M)$ can be written as follows :

$$\begin{aligned} \lim_{n\to\infty} \hat{z}_n(M) &= \sum_{i=1}^{k^*} \frac{1}{M}\int_{i/k^*}^{1} p\,dp \\ &= \frac{1}{2M}k^* - \frac{1}{6M}k^* - \frac{3}{12M} - \frac{1}{12k^*M} \\ &\approx \frac{1}{3M}k^* - \frac{1}{4M}. \end{aligned}$$

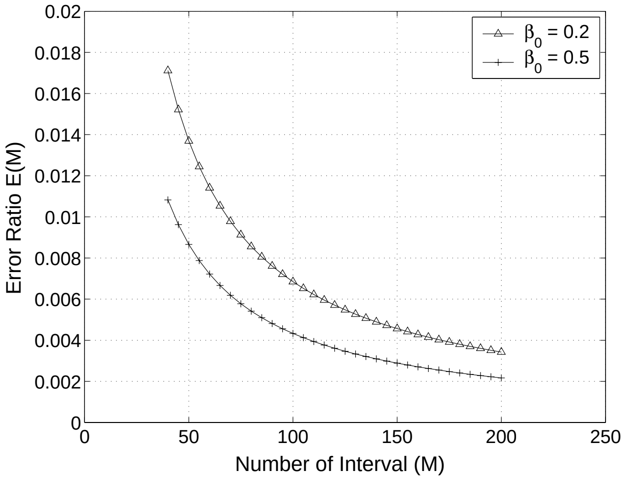

Figure 2: The error ratio $E(M)$ against the number of interval M.

Since $k^* \approx \sqrt{6\beta_0}\,M$,

$$\begin{aligned} \lim_{n\to\infty} \hat{z}_n(M) &\approx \sqrt{\frac{2\beta_0}{3}} - \frac{1}{4M}; \\ E(M) &\approx \frac{1}{4M}\sqrt{\frac{3}{2\beta_0}}. \end{aligned}$$

Equation (6) can thus be approximated by the following equation.

$$E(M) \approx \sqrt{\frac{3}{32}}\frac{h}{\sqrt{\beta_0}} = 0.3062\frac{1}{\sqrt{\beta_0}M}. \qquad (11)$$

Figure 2 shows the cases when β_0 equals to 0.2 and 0.5 respectively. M is taken from 40 up since the approximation does not hold for small M. It is found that the percentage error is less than 1% even when the number of intervals is only 70.

4 Application in MUCA

In the illustrative example mentioned in the last section, we roughly estimated the average profit gain, $\lim_{n\to\infty}\lim_{M\to\infty}\hat{z}(M)$, as $\sqrt{\frac{2(0.5)}{3}}$. In accordance with Equation (11), the percentage error can thus be obtained.

$$E(M) = 0.3062\frac{1}{\sqrt{0.5}\ 1000} = 4.33\times 10^{-4}$$

which is less than 0.05%. A better estimation for the average profit gain can also be evaluated as follows :

$$\begin{aligned} \lim_{n\to\infty} \hat{z}(M=1000) &= \left(1 - 4.33\times 10^{-4}\right)\sqrt{\frac{2(0.5)}{3}} \\ &= 0.5771 \end{aligned}$$

Using Equation (11), we can determine the maximum integer of M such that the percentage error is less than certain threshold η.

$$M \approx 0.3062\frac{1}{\sqrt{\beta_0}\ \eta}.$$

For $\eta = 0.01$ and $\beta_0 = 0.2$, it can readily be shown that $M \approx 69$. For $\eta = 0.01$ and $\beta_0 = 0.5$, $M \approx 44$. For M equals to 100 and β_0 is in $[0.2, 0.5]$, the percentage error is less than 1%.

5 Conclusion

The average profit gain, Equation (4), for the unbounded knapsack problem with fractional sized objects has been derived. An extended result for the case that object size is discrete has been obtained. It has been shown that the average discrete-sized profit gain can be expressed in term of β_0 and M as $\sqrt{\frac{2\beta_0}{3}} - 0.3062(\sqrt{\beta_0}M)^{-1}$. An application of this result in multi-unit combinatorial auction has been presented. Without loss of generality, the case when p_i's are discrete random variables can also be derived using the same technique used in the paper.

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